QFT Beyond Fixed Order

Introduction to Bremsstrahlung and Jets

1. Radiation from Accelerated Charges

Soft Bremsstrahlung in Classical E&M, and in QED. ***The dipole factor*** *& coherence.*

2. Infrared Singularities and Infrared Safety

IR Poles & Sudakov Logarithms. ***Probabilities > 1.***

Summing over degenerate quantum states (KLN theorem). ***IRC Safety.***

3. QCD as a Weakly Coupled Conformal Field Theory

The ***emission*** *probability; Double-Logarithmic Approximation*

The ***no-emission*** *probability; Sudakov Factor; exponentiation; example:* ***jet mass****.*

4. Parton Showers

DLA as differential evolution kernels; unitarity and detailed balance.

Sampling the Sudakov; perturbation theory as a Markov Chain; Monte Carlo.

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Two Axioms for Infinite-Order Perturbative QCD

This lecture is based on **A. Larkoski**, *"An Unorthodox Introduction to QCD"*,:arXiv:1709.06195

1. At high energies, the coupling of QCD, α_s, is small.

$\Longrightarrow$ **QCD perturbation theory** (e.g., with Feynman diagrams) is a good approximation.

Sensible to describe final states in terms of quarks and gluons.

2. At high energies, QCD has no intrinsic scales.

QCD is (approximately) a conformal, or scale-invariant, quantum field theory:

Action integral for $\mathscr{L}_{\mathrm{QCD}}^{\mathrm{massless}} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + i\bar{\psi}_q \not{D}\psi_q$ invariant under scale transformations.

The strong coupling is (approximately) constant, independent of energy.

At (asymptotically) high energies, quark masses are negligible.

Strictly speaking, (2) is of course not really true.

There are (quark and hadron) mass scales in the theory, and the strong coupling runs.

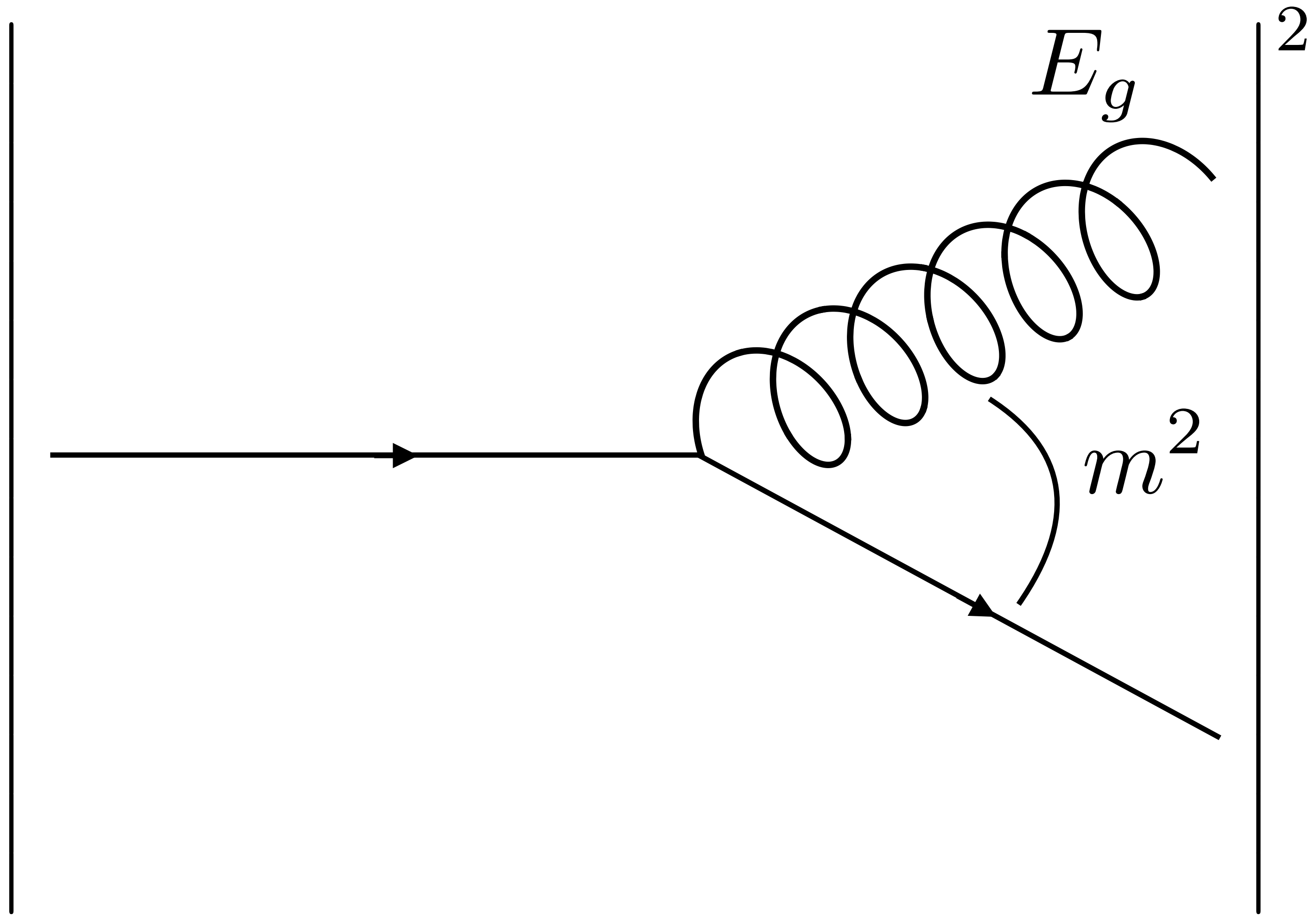
But the running is logarithmic (slow), and at energies above ~ 10 GeV only m_t is really large

➤ We will think of $\beta_{\mathrm{QCD}} \neq 0$ and $m_q \neq 0$ as **small corrections** on a **scale-invariant starting point**

Scale-invariant dynamics can only depend on **dimensionless** quantities (such as energy *ratios*, or angles)

In such a theory, what could be the allowed functional form of, say, the probability for a quark to emit a gluon?

$P(E_g, m^2)$ =



= ?

must be invariant if we "scale" all energies and masses by a factor, λ:

$\Longrightarrow$ Constraint Equation:

Note: we scale also the PS element

$$P(\lambda E_g, \lambda^2 m^2)\, d(\lambda E_g)\, d(\lambda^2 m^2) = P(E_g, m^2)\, dE_g\, dm^2$$

Q: why λ^2 for this argument?

Scale invariance

Simple Guess

What sort of functions fulfil **Scale invariance** **?**

$$P(\lambda E_g, \lambda^2 m^2)\, d(\lambda E_g)\, d(\lambda^2 m^2) = P(E_g, m^2)\, dE_g\, dm^2$$

$\propto$ $\mathrm{d}E/E$ and $\mathrm{d}m^2/m^2$

Dimensionless function of E_g/m with soft limit $f(0) \to 1$

$\Longrightarrow$ Simplest guess we can write down is:

$$P(E_g, m^2)\, dE_g\, dm^2 = N \frac{dE_g}{E_g} \frac{dm^2}{m^2} \times f(E_g^2/m^2)$$

Dimensionless normalisation constant.

Cannot fix this from scale invariance alone. For $q \to qg$, it must be proportional to $g_s^2 = 4\pi\alpha_s$, times some "Colour Charge" = C_F = 4/3 for an SU(3) triplet. The $1/\pi$ is chosen to produce the known expressions in QCD (such as the dipole factor).

The DLA Emission Probability

The "double-logarithmic approximation" (DLA) is obtained via the soft limit $f \to 1$

Express $\mathrm{d}E_g\,\mathrm{d}m^2_{qg}$ in terms of $\mathrm{d}z_g\,\mathrm{d}(\cos\theta_{qg})$ with

$$z_g = \frac{E_g}{E_g + E_q}$$

$$m^2_{qg} = E_g E_q (1 - \cos\theta_{qg})$$

$$P(z, \cos\theta)\,dz\,d\cos\theta = \frac{\alpha_s C_F}{\pi}\frac{dz}{z}\frac{d\cos\theta}{1-\cos\theta}$$

(Compare with the expression for the soft-photon probability density we got using Feynman diagrams in the previous lecture:

$$\mathrm{d}P_\gamma = \frac{e^2}{4\pi^2}\frac{\mathrm{d}k}{k}\frac{\mathrm{d}\cos\theta_k}{(1-\cos\theta_k)}$$
)

Most Singular Limit: Simultaneously Soft **and** Collinear

Taking also the small-angle limit $\theta_{qg} \ll 1$

$1 - \cos\theta_{qg} \sim \theta_{qg}^2/2$

$$P(z,\theta^2)\,dz\,d\theta^2 \to \frac{\alpha_s C_F}{\pi}\frac{dz}{z}\frac{d\theta^2}{\theta^2}$$

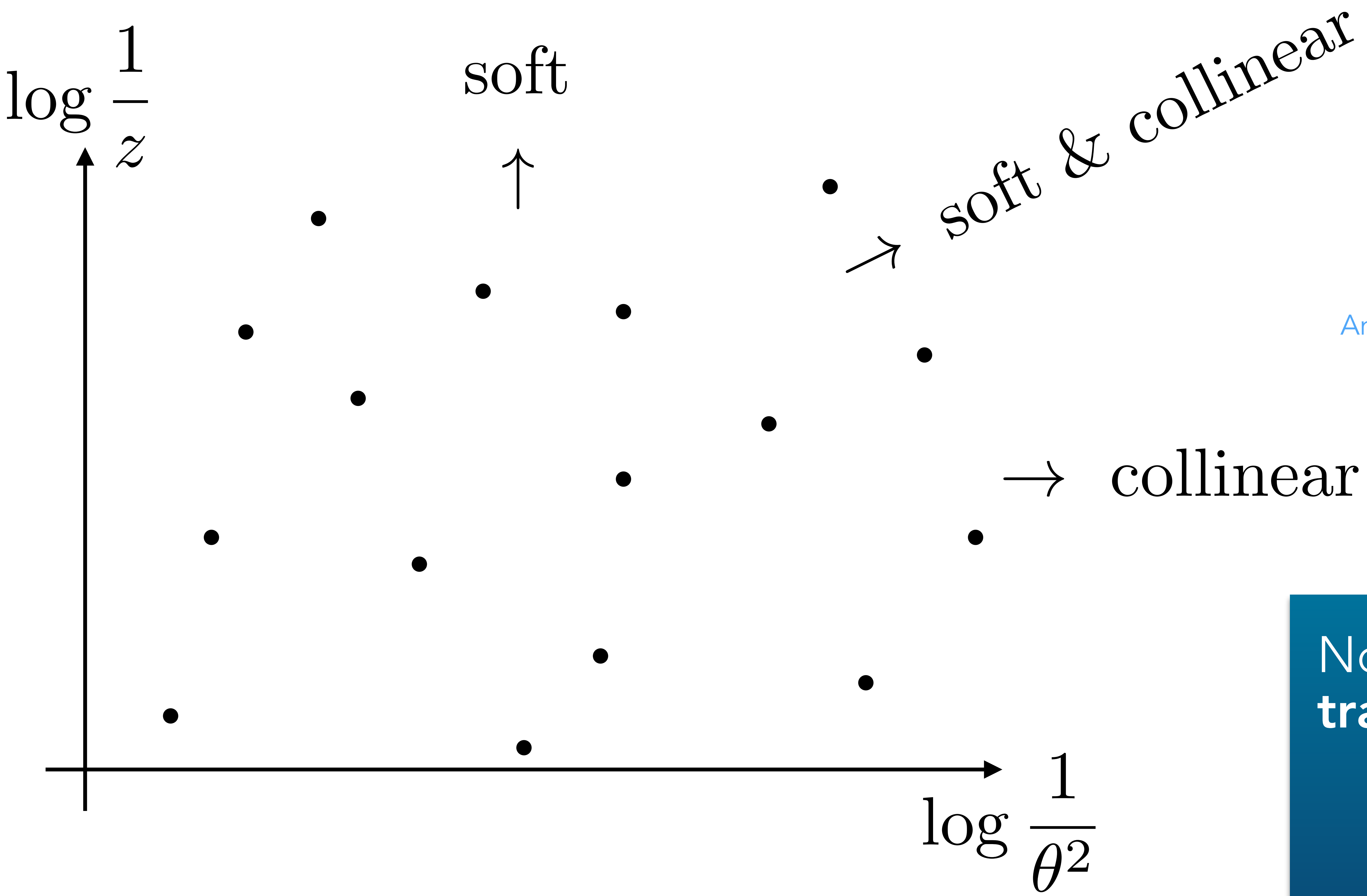
As discussed in the previous lecture, we should not interpret this as the probability to emit a single gluon (or photon), but rather as an expectation value for the **average number density** of emitted quanta.

Noting that the derivatives are of the form $\mathrm{d}x/x = \mathrm{d}(\log x)$, we rewrite:

$$P(z,\theta^2)\,dz\,d\theta^2 = \frac{\alpha_s C_F}{\pi}\frac{dz}{z}\frac{d\theta^2}{\theta^2} = \frac{\alpha_s C_F}{\pi}\,d\left(\log\frac{1}{z}\right)\,d\left(\log\frac{1}{\theta^2}\right)$$

- **A uniform distribution in ln(1/z) and ln(1/θ):**

$$P(z,\theta^2)\,dz\,d\theta^2 = \frac{\alpha_s C_F}{\pi}\frac{dz}{z}\frac{d\theta^2}{\theta^2} = \frac{\alpha_s C_F}{\pi}\,d\left(\log\frac{1}{z}\right)\,d\left(\log\frac{1}{\theta^2}\right)$$



Emissions **uniformly** distributed in the (dimensionless) **"Lund plane"**

Andersson, Gustafson, Lönnblad, Pettersson, Z. Phys C**43**(1989)625

Note: original **Lund plane** uses **transverse momentum** p_T and **rapidity**

$$\ln(p_{\perp g}/m_0) \sim \ln(z\theta)$$
$$y = -\ln\tan\theta/2$$

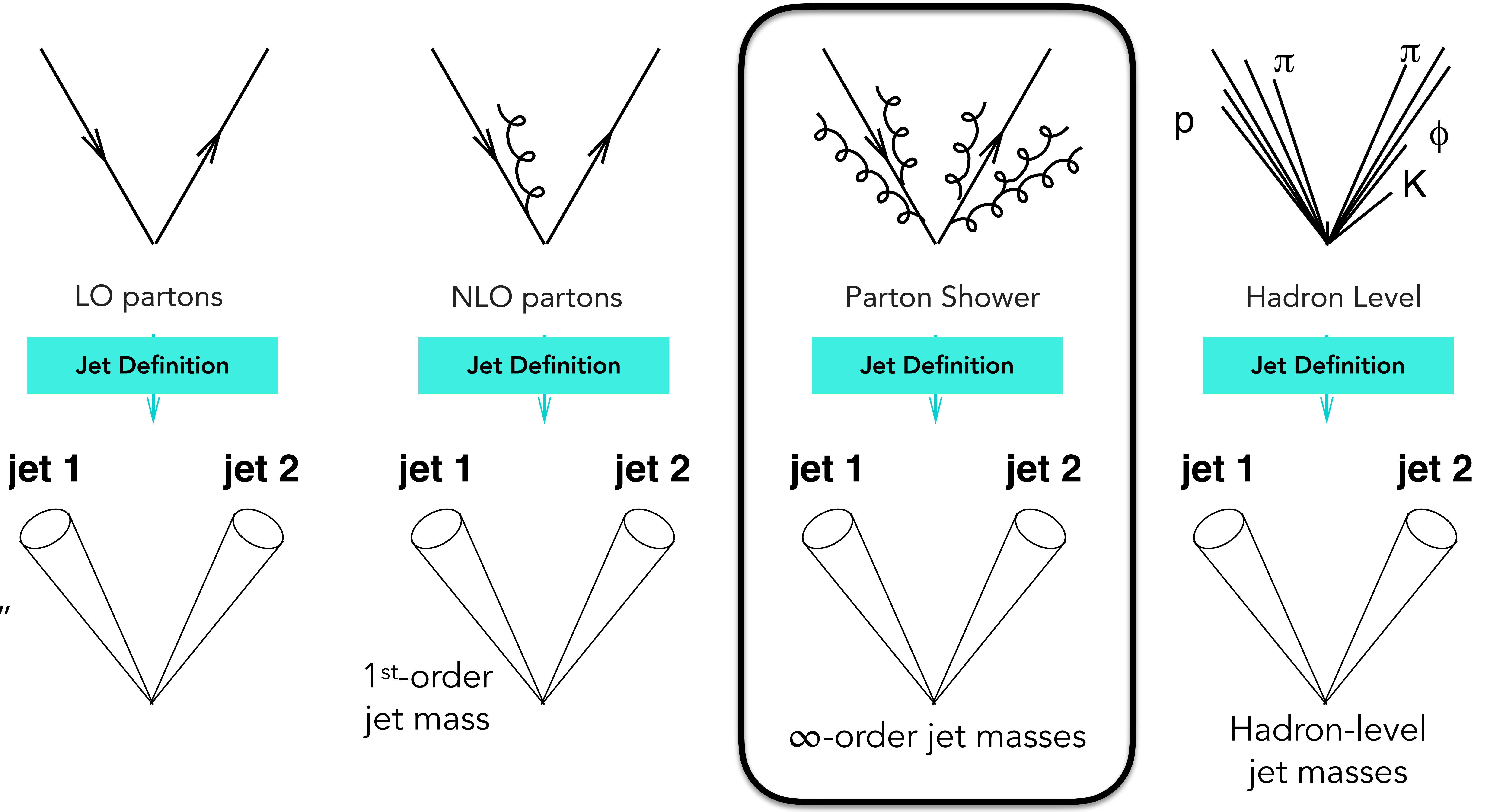
Practical Example: The invariant mass of a Jet

Let's **apply** our notion of a scale-invariant **uniform density** of emitted gluons in the log-log Lund plane to compute something real: **the invariant mass of a jet**, to **∞ perturbative order**

This calculation will of course only be **accurate** within the context of the double-log ~ **classical** (aka eikonal) approximation (DLA); should capture at least the **"most important" bremsstrahlung corrections**.

Think of jets as **projections** that provide a universal view of events

Illustrations by G. Salam

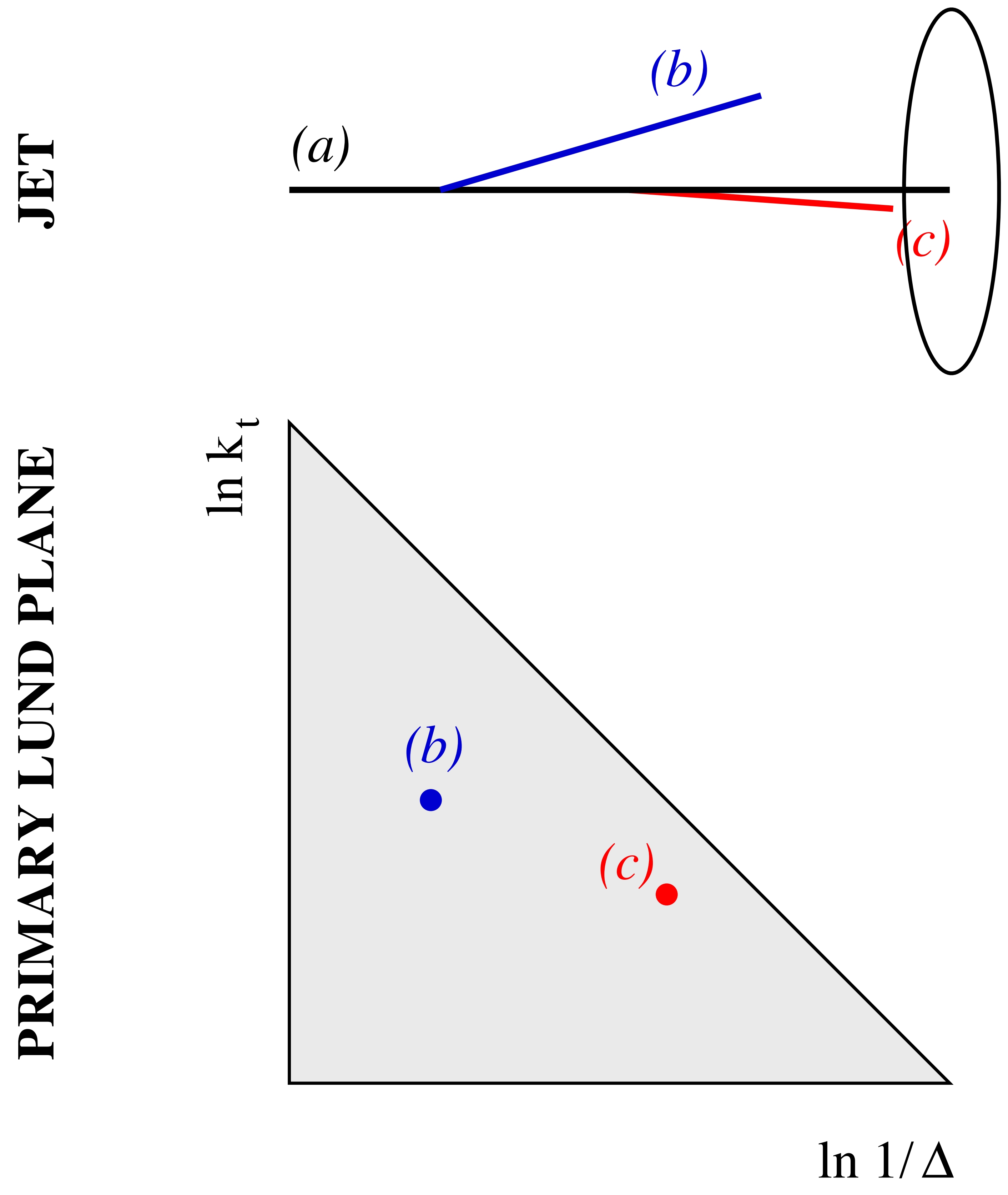


At LO, the "jet" mass is zero

(Note: details of different **types** of jet definitions & clustering algorithms (k_T, anti-k_T, C/A, cones, …) not covered here.

See e.g., lectures & notes by G. Salam.)

Dimensionless jet-massy observable



$$m^2 = (p_a + p_b + p_c)^2$$

$$= 2E_aE_b(1 - \cos\theta_{ab}) + 2E_bE_c(1 - \cos\theta_{bc}) + 2E_aE_c(1 - \cos\theta_{ac})$$

$$\to E_a^2\left(z_b\theta_b^2 + z_c\theta_c^2 + \mathcal{O}(z^2)\right)$$

$$\Longrightarrow \quad \tau = \frac{m_{\rm jet}^2}{E_{\rm jet}^2} \to \sum_i z_i\theta_i^2$$

(a.k.a. "1-Thrust")

The sum runs over all emitted gluons in the plane

Want to compute the probability to observe $\tau \leq \tau_{\rm cut}$

I.e.: what fraction of events will **survive** a cut requiring $m_{\rm jet}^2 \leq E_{\rm jet}^2\tau_{\rm cut}$?

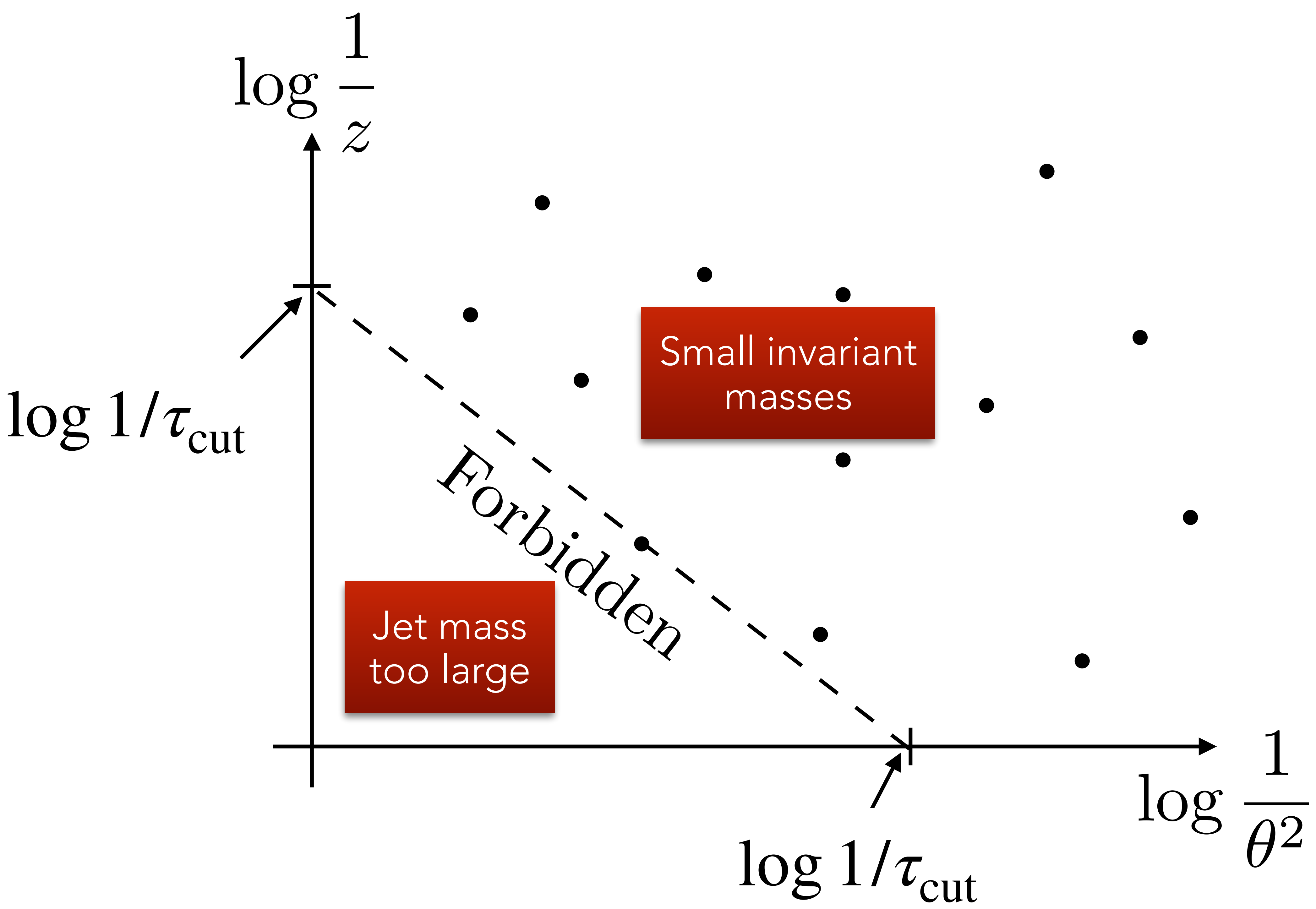
Equivalently what fraction of events will **fail** a cut requiring $m_{\rm jet}^2 \geq E_{\rm jet}^2\tau_{\rm cut}$?

τ is a sum of positive terms

For $\tau \leq \tau_{\text{cut}}$: no single term is allowed to be greater than τ_{cut}

One emission dominates:

Uniform log-log density $\Longrightarrow$ emissions exponentially far apart in (z, θ^2) $\Longrightarrow$ unlikely for event with $\max(\tau_i) < \tau_{\text{cut}}$ to get across the line



$\Longrightarrow$ Just compute probability for no emission in forbidden region

Caution: an event with two (or more) emissions in forbidden region can only be rejected **once** $\Longrightarrow$ Not just a simple integral of uniform density over that region.

The No-Emission Probability

To compute P(no emission), Larkoski splits up phase space in small subregions and multiplies together probabilities for no emission in any one of them (see backup slides)

Simpler to use our interpretation of integrated emission probability as average number of emissions (cf last lecture):

If the emissions are equivalent and independent (fine in our soft limit), we can interpret the average number of emissions in forbidden region:

$$\langle n \rangle(\tau_{\text{cut}}) = \frac{\alpha_s C_F}{2\pi} \log^2 \tau_{\text{cut}}$$

Average number of emissions with $\tau > \tau_{\text{cut}}$
= density times area of region with $\tau > \tau_{\text{cut}}$

as the mean of a Poisson distribution:

$$P(n) = \frac{\langle n \rangle^n \exp\left(-\langle n \rangle\right)}{n!}$$

Probability to have n emissions with $\tau > \tau_{\text{cut}}$

Hence the probability for no emissions in the requested region is $P(0)$:

Probability for no emissions with $\tau > \tau_{\text{cut}}$ (in Poissonian limit)

$$P(0) = \exp\left(-\frac{\alpha_s C_F}{2\pi} \log^2 \tau_{\text{cut}}\right)$$

Called the Sudakov "Form" Factor

(This is the same expression as Larkoski gets.)

The (all-orders) Emission Probability

To find the **probability distribution** to observe a given value of τ (i.e., the jet mass distribution), differentiate the no-branching probability wrt τ:

$$p(\tau) = \frac{d}{d\tau}\exp\left[-\frac{\alpha_s}{\pi}\frac{C_F}{2}\log^2\tau\right] = \boxed{-\frac{\alpha_s C_F}{\pi}\frac{\log\tau}{\tau}\exp\left[-\frac{\alpha_s}{\pi}\frac{C_F}{2}\log^2\tau\right]}\,.$$

Simple Interpretation: the differential rate of change of the **no-emission** probability is equal to (minus) the rate of emissions.

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There is a close analogy with the simple process of nuclear decay.
There the naive decay rate per unit time is given by the decay constant.
But a nucleus can only decay at a given time t if it has not already decayed.
The actual decay rate per nucleus in a sample is therefore c * exp(- c Δt).
Exercise: identify what plays the role of c, t, dt, and Δt, in our case.
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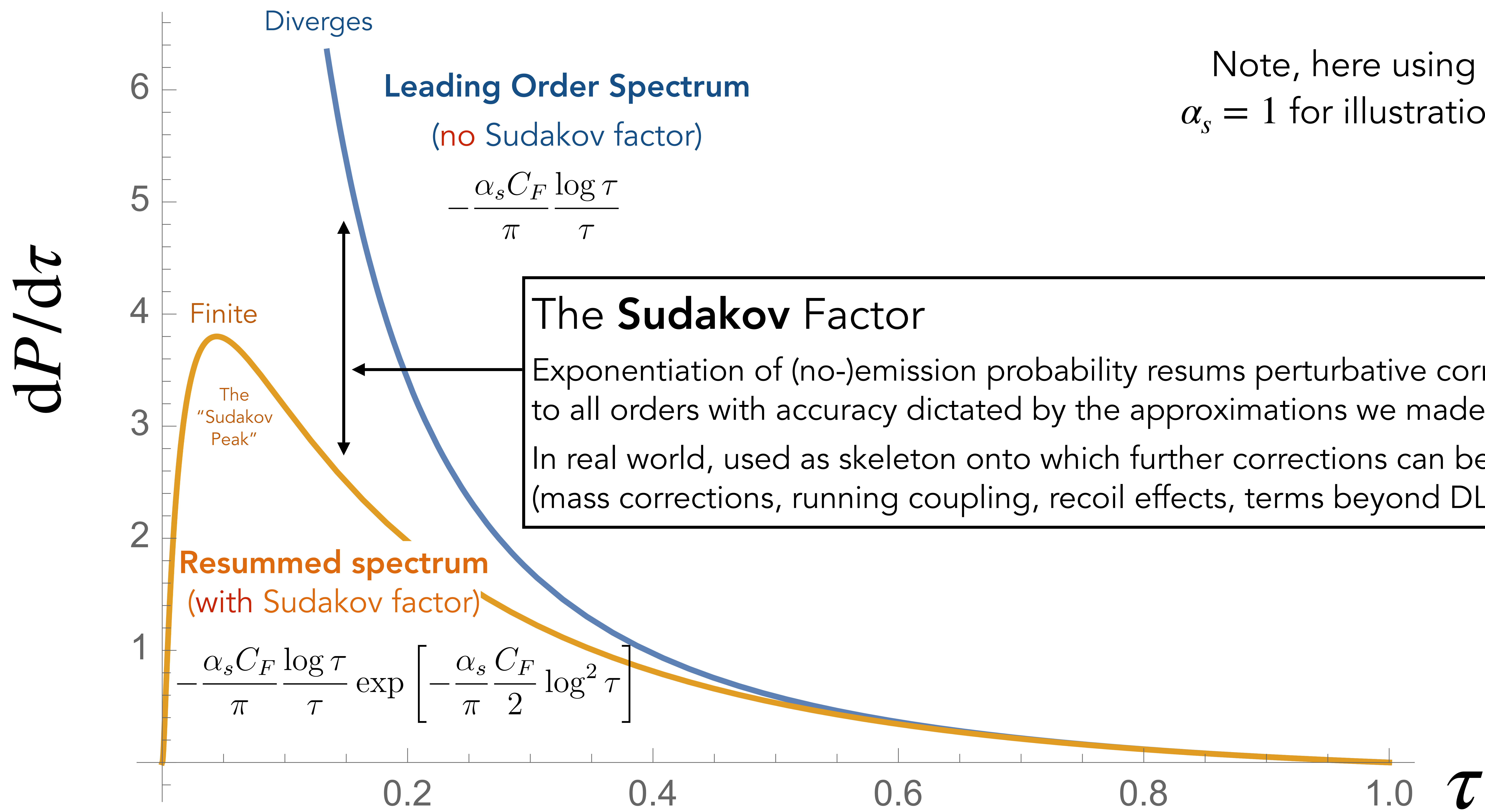
The Resummed Jet Mass Distribution



Note, here using $\alpha_s = 1$ for illustration

The Sudakov Factor

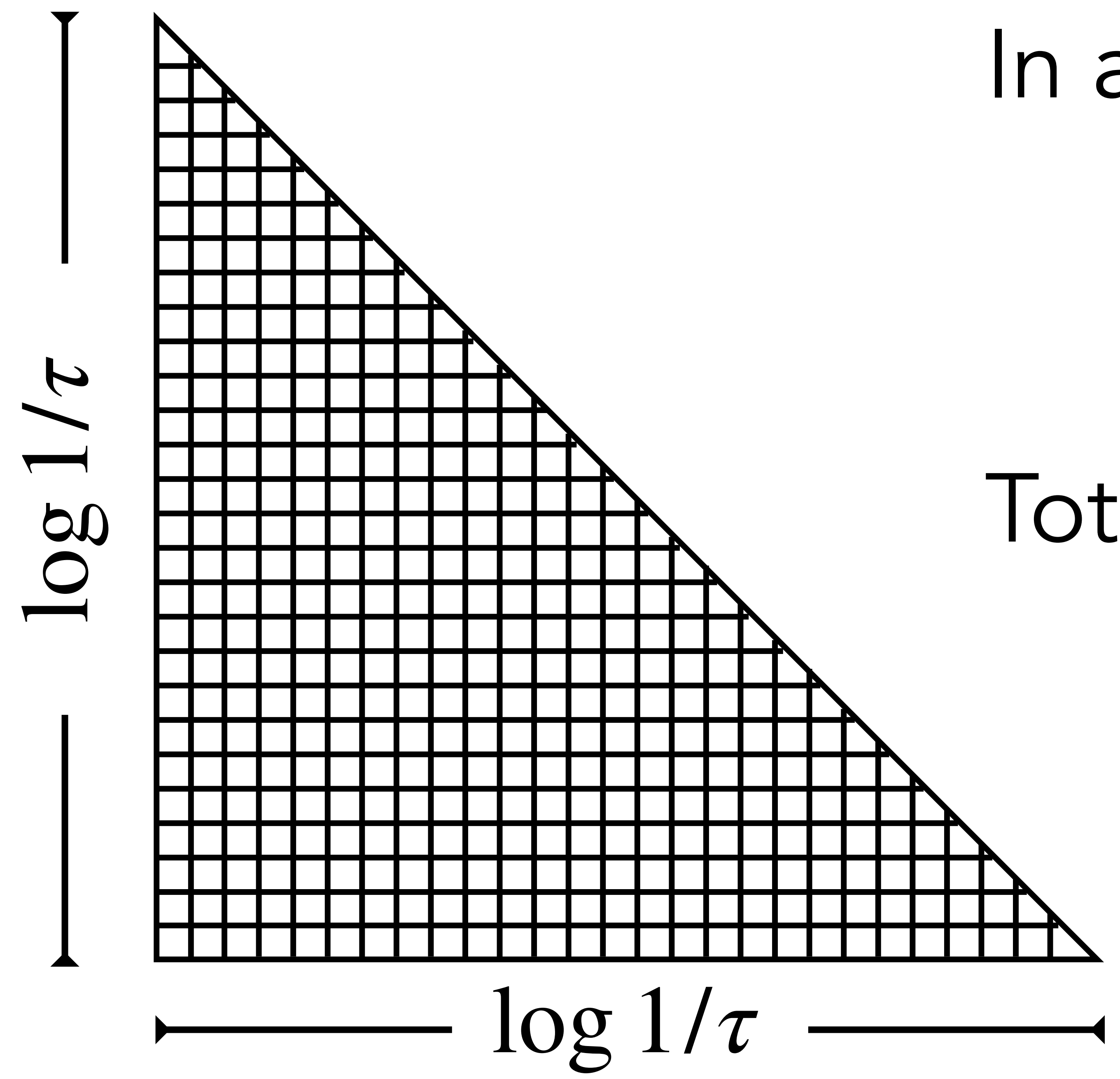
Exponentiation of (no-)emission probability resums perturbative corrections to all orders with accuracy dictated by the approximations we made.

In real world, used as skeleton onto which further corrections can be imposed (mass corrections, running coupling, recoil effects, terms beyond DLA, …)

NB: the jet mass distribution is of course just one example. Sudakov suppression (and the Sudakov peak) is characteristic for any distribution which is IR divergent at fixed order.

Extra Slides

Break up the forbidden area in tiny (differential) subregions:



In any one subregion, *i*, the probability for no emission is

$$P(\text{no emit in region } i) = 1 - \frac{\alpha_s C_F}{\pi} \cdot (\text{Area of region } i)\,.$$

Total area = $\frac{1}{2}\log^2\tau$ $\implies$ $\text{Area of region } i = \frac{\frac{1}{2}\log^2\tau}{N}$

No emission in any of these regions:

$$P(\text{no emissions}) = \left(1 - \frac{\frac{\alpha_s}{\pi}\frac{C_F}{2}\log^2\tau}{N}\right)^N \to \exp\left[-\frac{\alpha_s}{\pi}\frac{C_F}{2}\log^2\tau\right]$$

The Sudakov "Form" Factor

To find the **probability** **distribution** to observe a given value of τ (i.e., the jet mass distribution), differentiate the cumulative distribution wrt τ:

$$p(\tau) = \frac{d}{d\tau}\exp\left[-\frac{\alpha_s}{\pi}\frac{C_F}{2}\log^2\tau\right] = -\frac{\alpha_s C_F}{\pi}\frac{\log\tau}{\tau}\exp\left[-\frac{\alpha_s}{\pi}\frac{C_F}{2}\log^2\tau\right].$$