QCD and Event Generators

Lecture 1 of 3

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Disclaimer

This course covers:

i.e., fixed perturbative order in α_s : LO, NLO, …

Lecture 1: QCD at **Fixed Order**

Lecture 2: Beyond Fixed Order — **Showers** and Merging

Lecture 3: Beyond Perturbations — **Hadronization** and Underlying Event

Supporting Lecture Notes (~80 pages): *"Introduction to QCD"*, arXiv:1207.2389

+ MCnet Review: *"General-Purpose Event Generators"*, Phys.Rept.504(2011)145

It does not cover:

Jet Physics → *Lectures by A. Larkoski*
Resummation techniques other than showers
Simulation of BSM physics
Event Generator Tuning
Monte Carlo (sampling) techniques
Heavy Ions and Cosmic Rays
+ many other (more specialised) topics such as: heavy quarks, hadron and τ decays, exotic hadrons, lattice QCD, loop amplitude calculations, spin/polarisation, non-global logs, subleading colour, factorisation caveats, PDF uncertainties, DIS, low-x, low-energy, higher twist, pomerons, rescattering, coalescence, neutrino beams, …

Plenty more could be said about QCD.

Focus here is on "users of QCD"


Q C D

$$\mathscr{L} = \bar{q}^i_\alpha (i\gamma^\mu)_{\alpha\beta} (D_\mu)^{ij}_{\beta\delta} q^j_\delta - m_q \bar{q}^i_\alpha q^i_\alpha - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

$i, j \in [1,3]$: fundamental-rep SU(3) **colour** indices
$a \in [1,8]$: adjoint-rep SU(3) **colour** index
$\alpha, \beta, \ldots \in [1,4]$: Dirac **spinor** indices

Quark fields

$$\psi^j_q = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

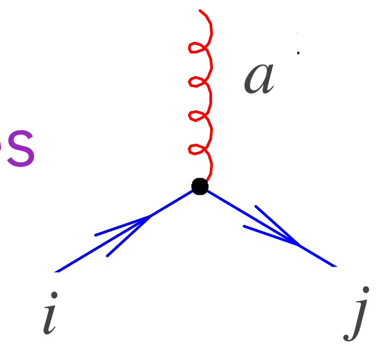
L invariant under $\psi \to U\psi$

SU(3) Local Gauge Symmetry

Gluon Gauge Fields & Covariant Derivative

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu - i g_s t^a_{ij} A^a_\mu$$

$\Rightarrow$ Feynman rules



$a \in [1,8]$

$i, j \in [1,3]$

with the **Gell-Mann Matrices** $(t^a = \tfrac{1}{2}\lambda^a)$

(Traceless and Hermitian)

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda^8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{-2}{\sqrt{3}} \end{pmatrix}$$

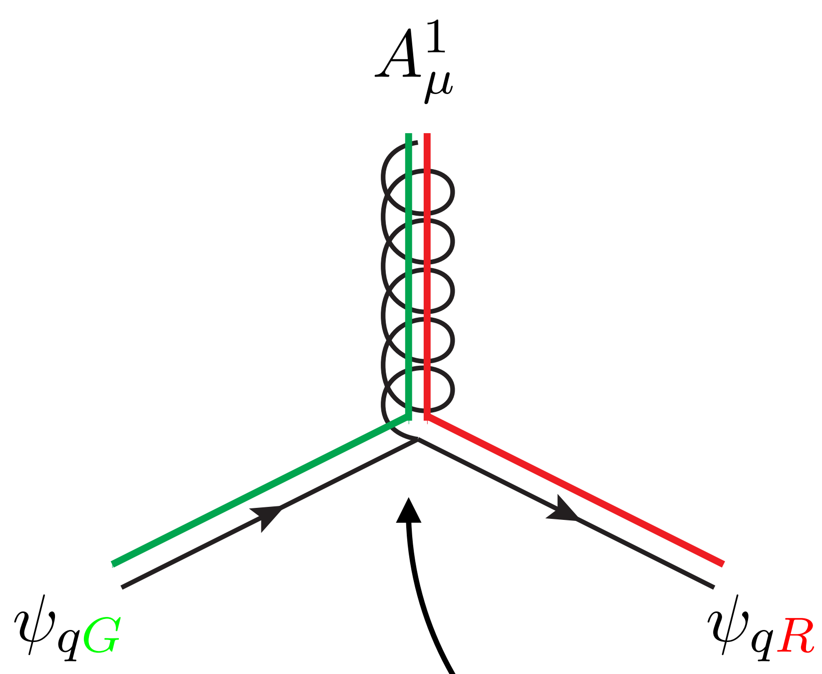
Interactions in Colour Space

A quark-gluon interaction

(= one term in sum over colours)

$$\mathscr{L} \ : \quad \bar{\psi}_q^i (i\gamma^\mu)(D_\mu)_{ij}\psi_q^j$$

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu - i g_s t^a_{ij} A^a_\mu$$



$$\propto \quad -\tfrac{i}{2} g_s \quad \bar{\psi}_{qR} \quad \lambda^1 \quad \psi_{qG}$$

$$= \quad -\tfrac{i}{2} g_s \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Gluon (adjoint) colour index ∊ [1,8]

Gluon Lorentz-vector index ∊ [0,3]

$$-i\, g_s\, t^1_{ij}\, \gamma^\mu_{\alpha\beta}\, A^1_\mu \quad -i\, g_s\, t^2_{ij}\, \gamma^\mu_{\alpha\beta}\, A^2_\mu \quad - \ldots$$

Quark colour indices ∊ [1,3]

Fermion spinor indices ∊ [1,4]

Amplitudes Squared **summed over colours** → traces over products of *t* matrices
→ ***Colour Factors*** (see e.g. lecture notes & backup slides)

Gluons are (colour) **charged**

This is a signature of any **non-Abelian** gauge theory

Non-commuting generators; matrix-valued vertices

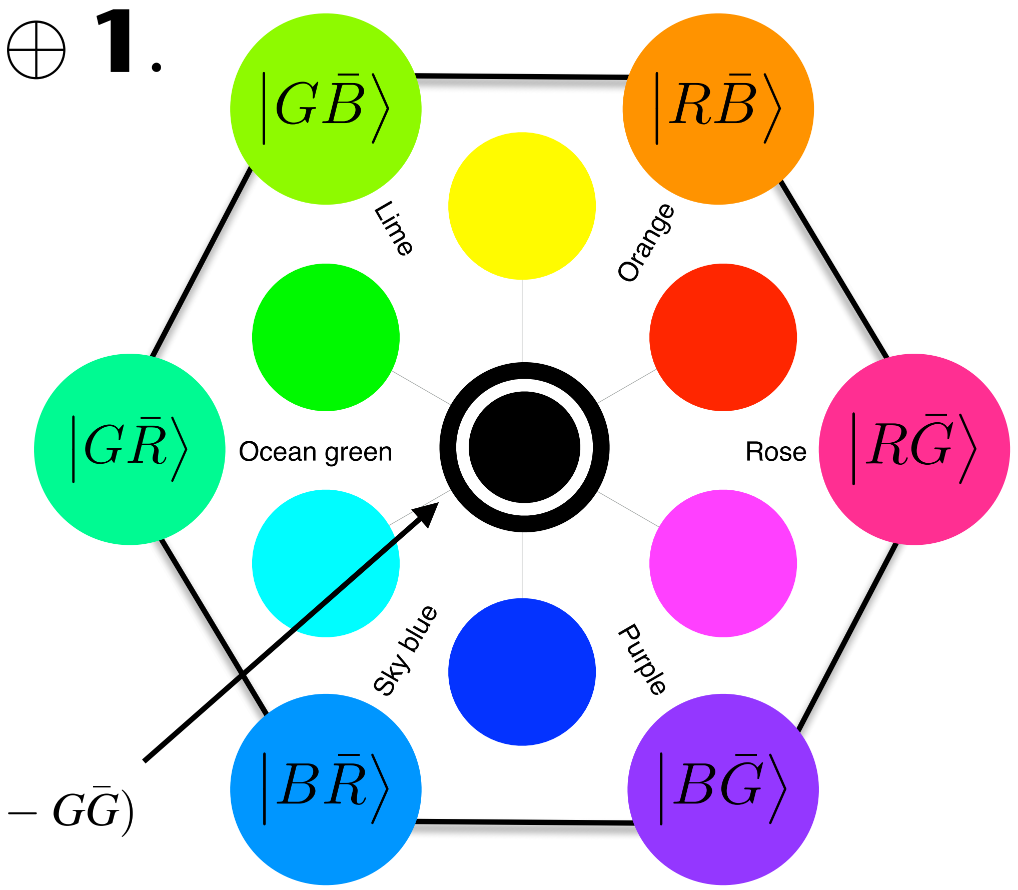
Gluons represent (matrix) transformations in colour space, which "repaint" quarks

One way of representing the octet is via $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$.

Under SU(3) transformations, these states transform into each other, but never go "outside" the multiplet.

(Like the S_z value of a particle with a certain spin changes under rotations, but its total spin does not.)

Note in the standard rep, the GM matrices are cast as linear combinations of these e.g. $\lambda^1 = (R\bar{G} + G\bar{R})$



$$g_7 = \frac{1}{\sqrt{2}}(R\bar{R} - G\bar{G})$$

$$g_8 = \frac{1}{\sqrt{6}}(R\bar{R} + G\bar{G} - 2B\bar{B}).$$

(The two states in the middle correspond to "m=0" components)
(We say they generate the U(1)² "Cartan subalgebra" of SU(3))

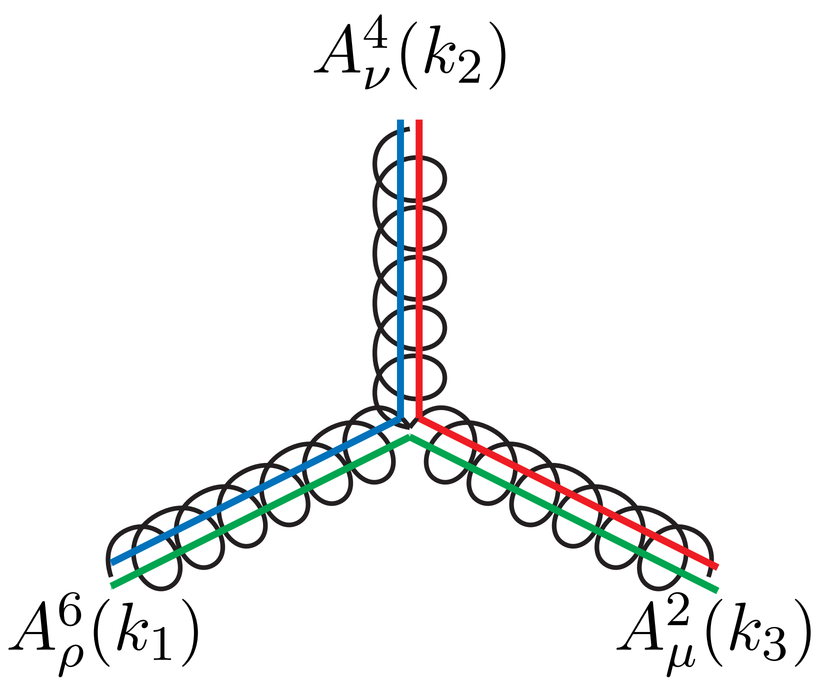
Interactions in Colour Space: Gluon Self-Interactions

A gluon-gluon interaction

(no equivalent in QED)

$$\mathscr{L} : \quad -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

$$F^a_{\mu\nu} = \underbrace{\partial_\mu A^a_\nu - \partial_\nu A^a_\mu}_{\text{Abelian}} + \underbrace{g_s f^{abc} A^b_\mu A^c_\nu}_{\text{non-Abelian}}$$



$$\propto \quad -g_s\, f^{246} \quad [(k_3 - k_2)^\rho g^{\mu\nu} + (k_2 - k_1)^\mu g^{\nu\rho} + (k_1 - k_3)^\nu g^{\rho\mu}]$$

$$i f^{abc} = 2\mathrm{Tr}\{t^c [t^a, t^b]\}$$

Structure Constants of SU(3)

$$f_{123} = 1 \qquad (14)$$

$$f_{147} = f_{246} = f_{257} = f_{345} = \frac{1}{2} \qquad (15)$$

$$f_{156} = f_{367} = -\frac{1}{2} \qquad (16)$$

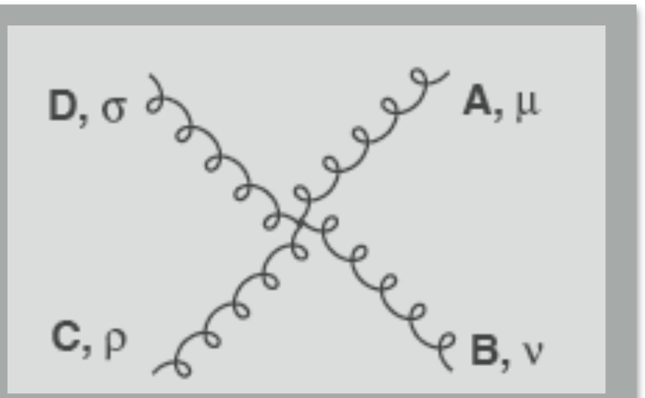
$$f_{458} = f_{678} = \frac{\sqrt{3}}{2} \qquad (17)$$

Antisymmetric in all indices

All other $f_{abc} = 0$

(Note there is also a 4-gluon vertex $\propto g_s^2$ with more complicated vertex factor

$$-ig_s^2 f^{XAC} f^{XBD} [g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\gamma}] + (C,\gamma) \leftrightarrow (D,\rho) + (B,\nu) \leftrightarrow (C,\gamma)$$



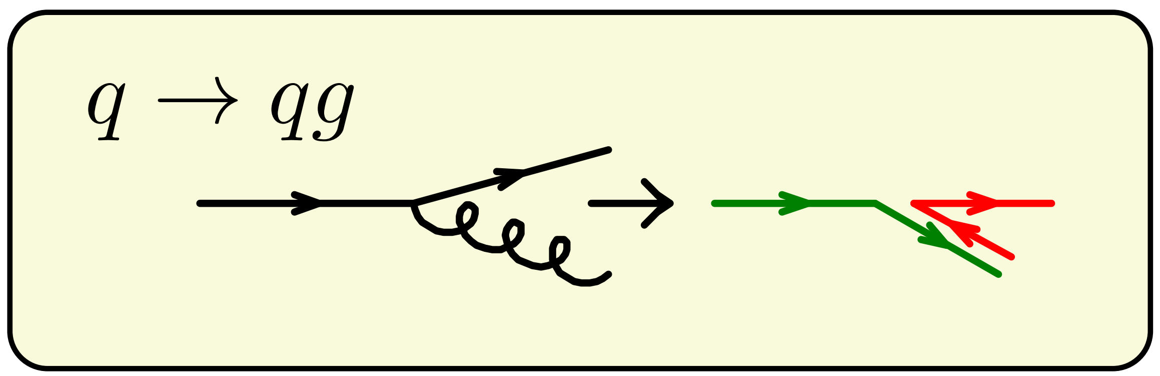
Note on Colour Vertices in Event Generators

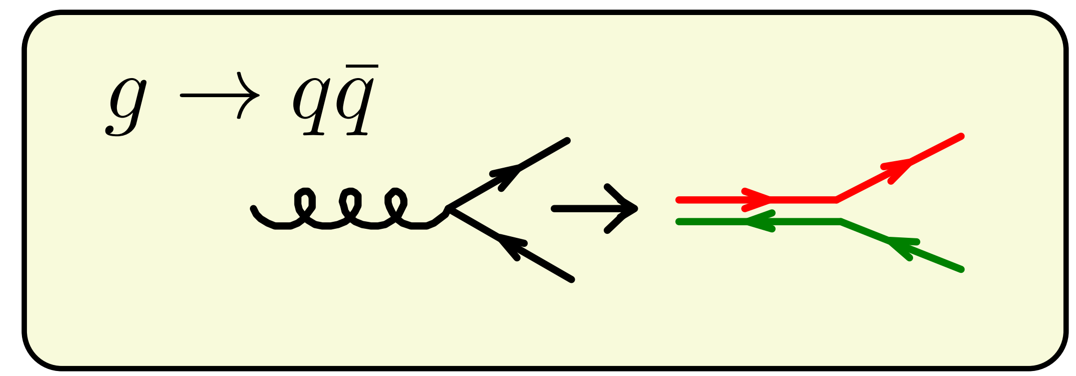
MC generators use a simple set of rules for "colour flow"

Based on **"Leading Colour"** (**LC**)

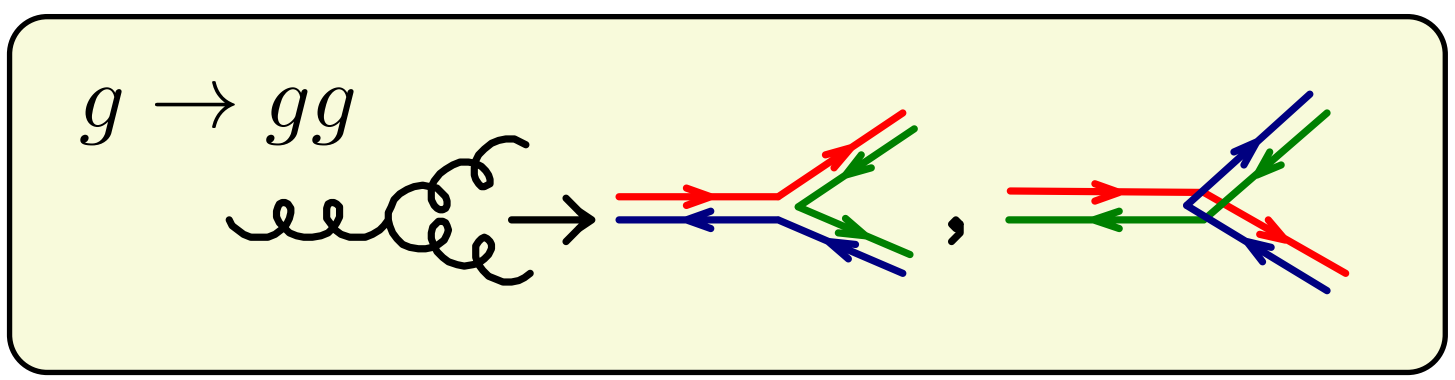
$$8 = \boxed{3 \otimes \overline{3}} \ominus 1$$

LC: gluons = outer products of colour and anticolour.
⇛ valid to ~ $1/N_C^2$ ~ 10% (exact in limit $N_C \to \infty$).





Illustrations from PDG Review on *MC Event Generators*



MCs: $N_C \to \infty$ limit formalised by letting **each "colour line"** be represented by a **unique Les Houches colour tag†** (no interference between different colour lines in this limit)

LC also used to assign Les Houches colour flows† in **hard processes**: $P_i = \dfrac{|M_i|^2}{\sum_{j \in LC} |M_j|^2}$

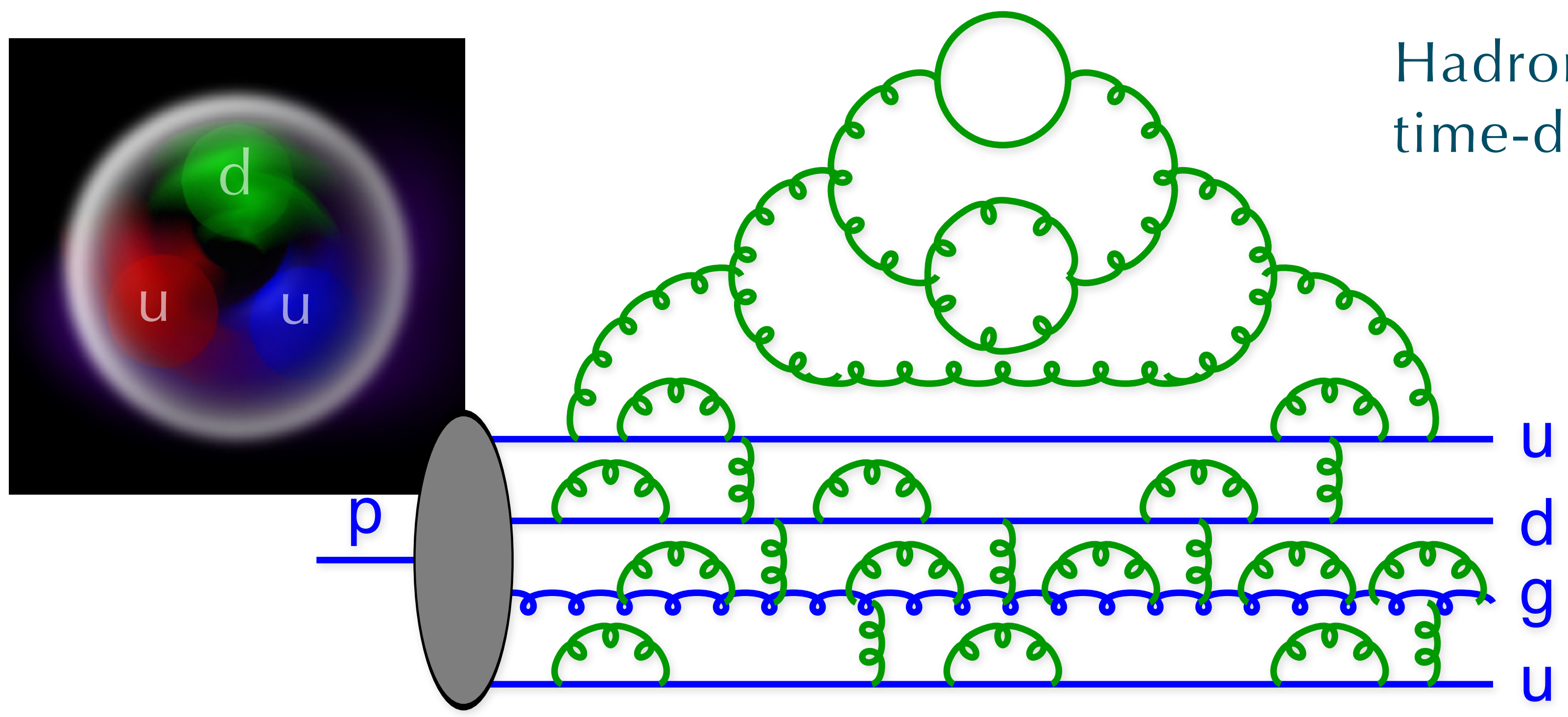
i.e., high-energy

†: hep-ph/0109068; hep-ph/0609017

Can we calculate LHC processes now?

What are we really colliding?

Take a look at the quantum level



Hadrons are composite, with time-dependent structure

(illustration by T. Sjöstrand)

Describe this mess **statistically** ➜ **parton distribution functions (PDFs)**

PDFs: $\boldsymbol{f_i(x,Q_F^2)}$ $\quad i \in [g,u,d,s,c,(b),(t),(\gamma)]$

Probability to find parton of flavour i with momentum fraction x,
as function of "resolution scale" Q_F ~ virtuality / inverse lifetime of fluctuation

Why PDFs work 1: heuristic explanation

Lifetime of typical fluctuation ~ r_p/c (=time it takes light to cross a proton)

~ 10^{-23} s; Corresponds to a frequency of ~ 500 billion THz

To the LHC, that's slow! (reaches "shutter speeds" thousands of times faster)

Planck-Einstein: $E=h\nu \to \nu_{LHC} = 13\ \mathrm{TeV}/h = 3.14$ million billion THz

➜ Protons look "frozen" at moment of collision

But they have a lot more than just two "u" quarks and a "d" inside

Difficult/impossible to calculate, so use statistics to parametrise the structure: **parton distribution functions** (PDFs)

Every so often I will pick a gluon, every so often a quark (antiquark)

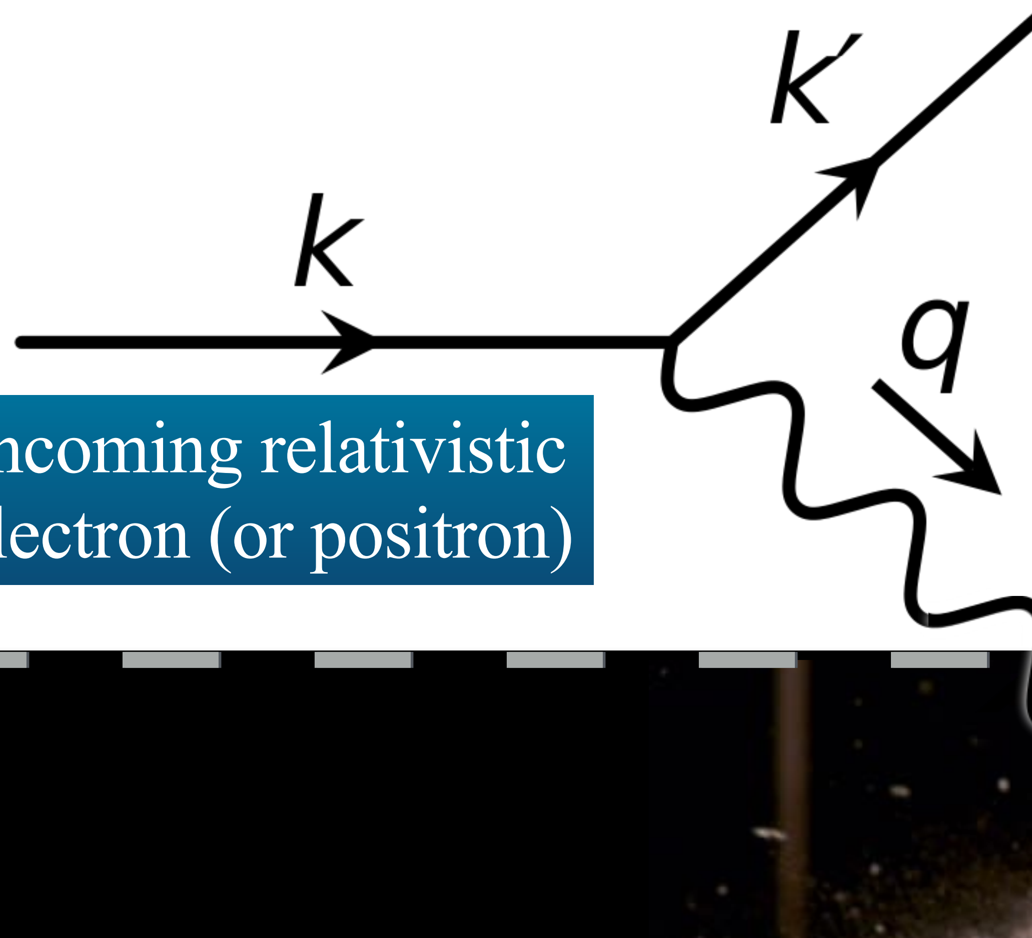
Measured at previous colliders (+ increasingly also at LHC)

Expressed as functions of energy fractions, x, and resolution scale, Q^2

+ obey known scaling laws $df\,/\,dQ^2$: "**DGLAP equations**".

Why PDFs work 2: Deep Inelastic Scattering

"Inelastic" = proton breaks up



Scattered electron

Incoming relativistic electron (or positron)

Hard (i.e. high-energy) photon
$q^2 = (k - k')^2 < 0$ (spacelike)
$\Longrightarrow$ often use $Q^2 \equiv -q^2 > 0$ instead

Leptonic part ~ **clean**

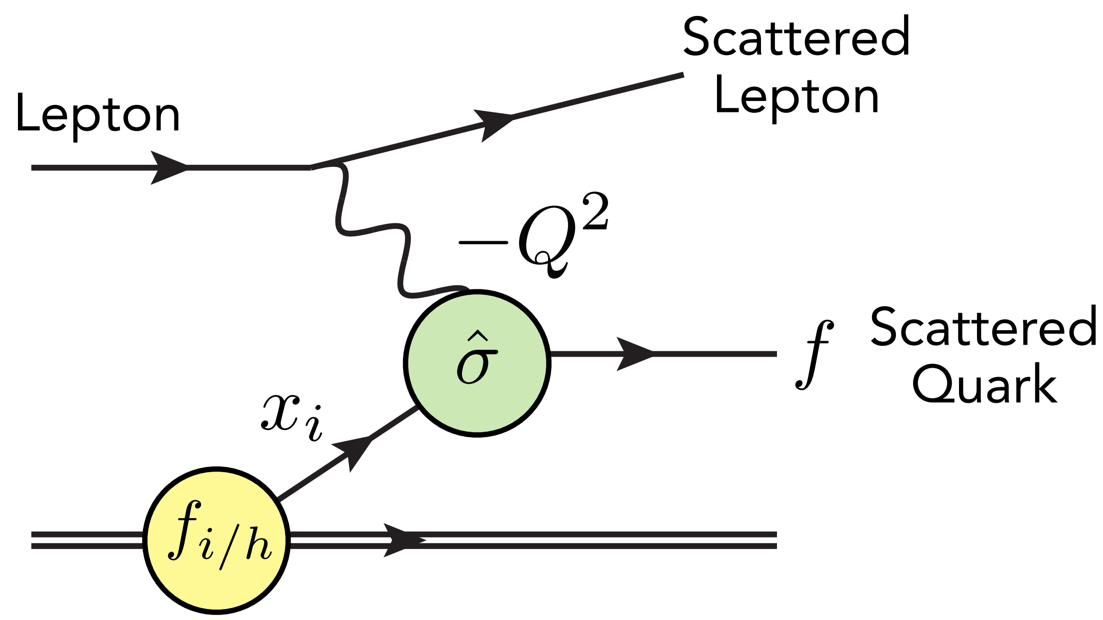
"Deep" = invariant mass of final hadronic system $\gg M_{proton}$

Hadronic part : **messy**

Why PDFs work 2: factorisation in DIS

Collins, Soper (1987): **Factorisation in Deep Inelastic Scattering**

Deep Inelastic Scattering (DIS)



We **assume*** that an analogous factorisation works for pp

*caveats are beyond the scope of this course

→ The cross section can be written in **factorised** form :

$$\sigma^{\ell h} = \sum_i \sum_f \int dx_i \int d\Phi_f \, f_{i/h}(x_i, Q_F^2) \, \frac{d\hat{\sigma}^{\ell i \to f}(x_i, \Phi_f, Q_F^2)}{dx_i \, d\Phi_f}$$

Sum over Initial (i) and final (f) parton flavors

Φ_f = Final-state phase space

$f_{i/h}$ = PDFs
Assumption: $Q^2 = Q_F^2$

Differential partonic **Hard-scattering** Matrix Element(s)
"hard" scale ~ Q^2

Factorisation $\Longrightarrow$ we can still calculate!

We're colliding, and observing, hadrons, but **can still do pQCD**

pQCD = perturbative QCD

PDFs: connect incoming hadrons with the high-scale process

Fragmentation Functions: connect high-scale process with final-state hadrons

Both combine **non-perturbative input** + all-orders (perturbative) bremsstrahlung **resummations**

$$\frac{\mathrm{d}\sigma}{\mathrm{d}X} = \sum_{a,b} \sum_{f} \int_{\hat{X}_f} f_a(x_a, Q_i^2) f_b(x_b, Q_i^2) \frac{\mathrm{d}\hat{\sigma}_{ab\to f}(x_a, x_b, f, Q_i^2, Q_f^2)}{\mathrm{d}\hat{X}_f} D(\hat{X}_f \to X, Q_i^2, Q_f^2)$$

PDFs: needed to compute inclusive cross sections

Hard Process
Fixed-Order QFT

FFs: needed to compute (semi-)exclusive cross sections

In MCs → **initial-state radiation** + non-perturbative hadron (beam-remnant) structure + multi-parton interactions

Matching & Merging

In MCs: **resonance decays + final-state radiation +** hadronisation + hadron decays (+ final-state interactions?)

Bjorken scaling:

If the strong coupling did not "run", QCD would be SCALE INVARIANT (a.k.a. conformal, e.g., N=4 Supersymmetric QCD)

Jets inside jets inside jets …

Loops inside loops inside loops …

Since α_s only **runs slowly (logarithmically)** $\Longrightarrow$ can still gain **all-orders** insight from scale-invariant properties ➜ **fractal** analogy for $Q \gg 1\,\mathrm{GeV}$ (→ lecture 2 on showers)

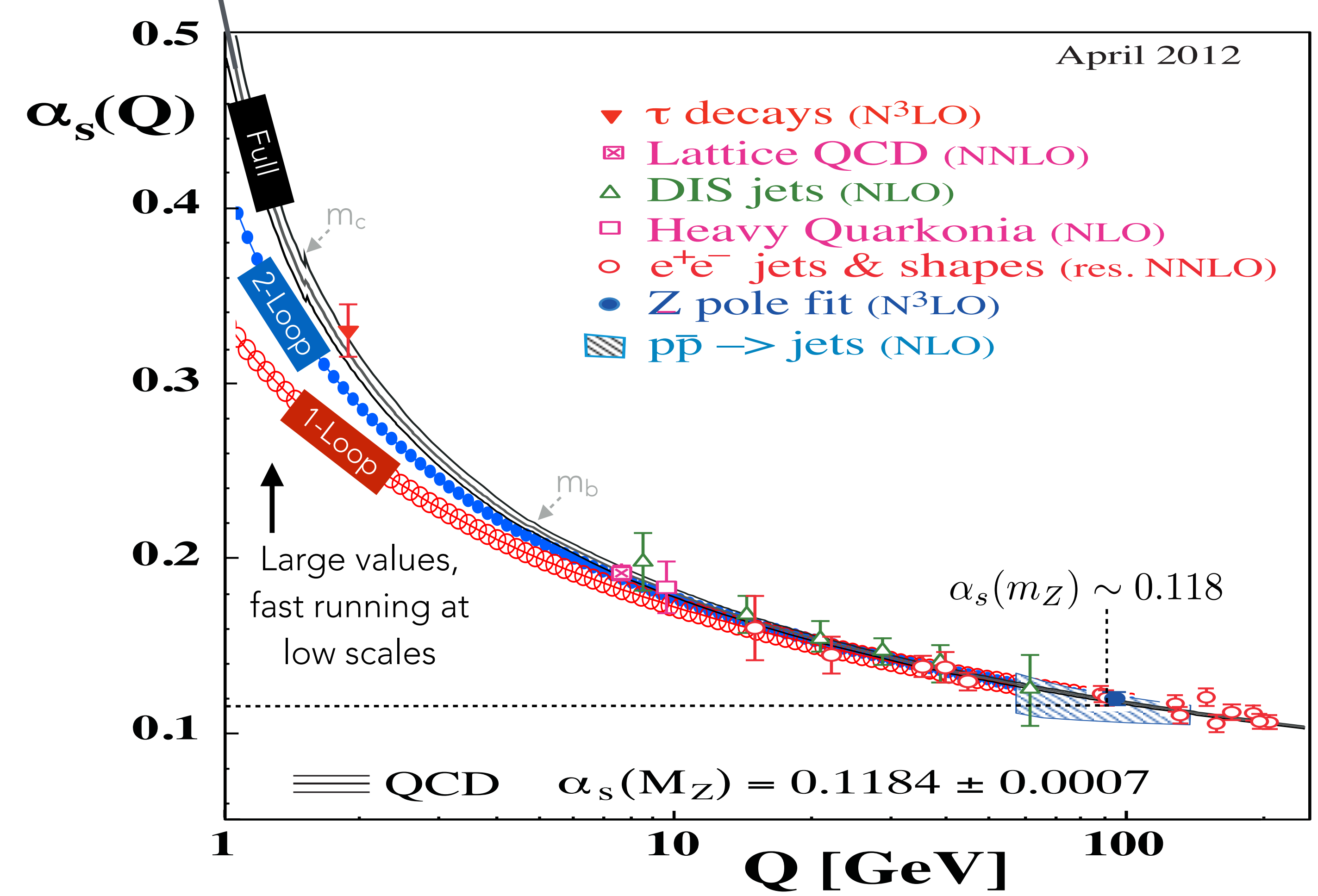
$$Q^2 \frac{\partial \alpha_s}{\partial Q^2} = \beta(\alpha_s)$$

$$= -\alpha_s^2(\boxed{b_0} + b_1\alpha_s + b_2\alpha_s^2 + \ldots)$$

Asymptotic Freedom

1-Loop β function coefficient: $b_0 = \frac{11C_A - 2n_f}{12\pi} > 0$ for $n_f \le 16$

Landau Pole at Λ_{QCD}~200 MeV



Note: I use the terms "conformal" and "scale invariant" interchangeably

Strictly speaking, conformal (angle-preserving) symmetry is more restrictive than just scale invariance

The Strong Coupling

$$\Longrightarrow \quad \alpha_s(Q^2) = \alpha_s(m_Z^2)\frac{1}{1 + b_0\ \alpha_s(m_Z) \ln \frac{Q^2}{m_Z^2} + \mathcal{O}(\alpha_s^2)} \qquad b_0 = \frac{11N_C - 2n_f}{12\pi}$$

The strong coupling is the **main parameter** of perturbative QCD calculations. It controls:

- The size of **QCD cross sections** (& QCD **partial widths** for decays).
- The overall amount of **QCD radiation** (extra jets + **recoil effects** + jet substructure).
- Sizeable **QCD "K Factors"** to essentially all processes at LHC, and **ditto uncertainties**.

Would like to have **reliable** (i.e., foolproof & exhaustive) way to **estimate QCD uncertainties**

In the absence of such a method, **variations of the renormalisation-scale argument** in α_s are widely used to estimate perturbative uncertainties; why?

$$\alpha_s(Q_1^2) - \alpha_s(Q_2^2) \ = \ \alpha_s^2\, b_0 \ln(Q_2^2/Q_1^2) \ + \ \mathcal{O}(\alpha_s^3)$$

→ Generates terms one order higher, proportional to what you already have (|M|²)

The (would-be) **all-orders** answer must be independent of our choice $\Longrightarrow$ uncalculated terms must **at least** contain same terms with opposite signs, to compensate

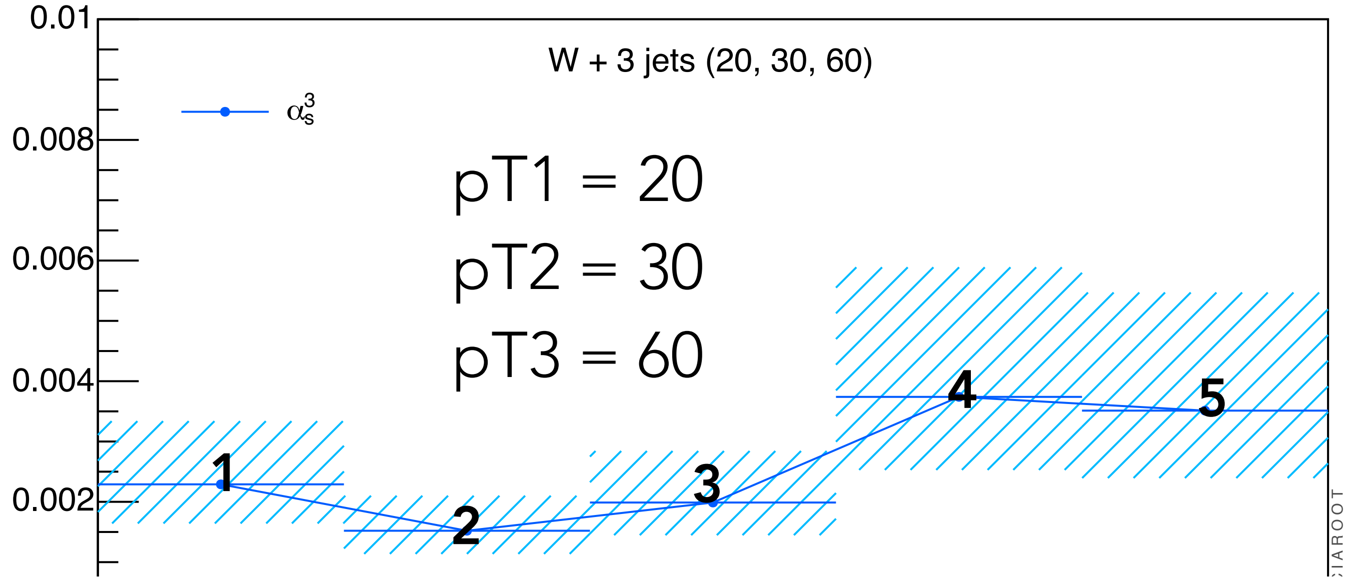
$\Longrightarrow$ a first **naive** way to estimate (lower bound on) uncertainty (more than beta function in rest of series).

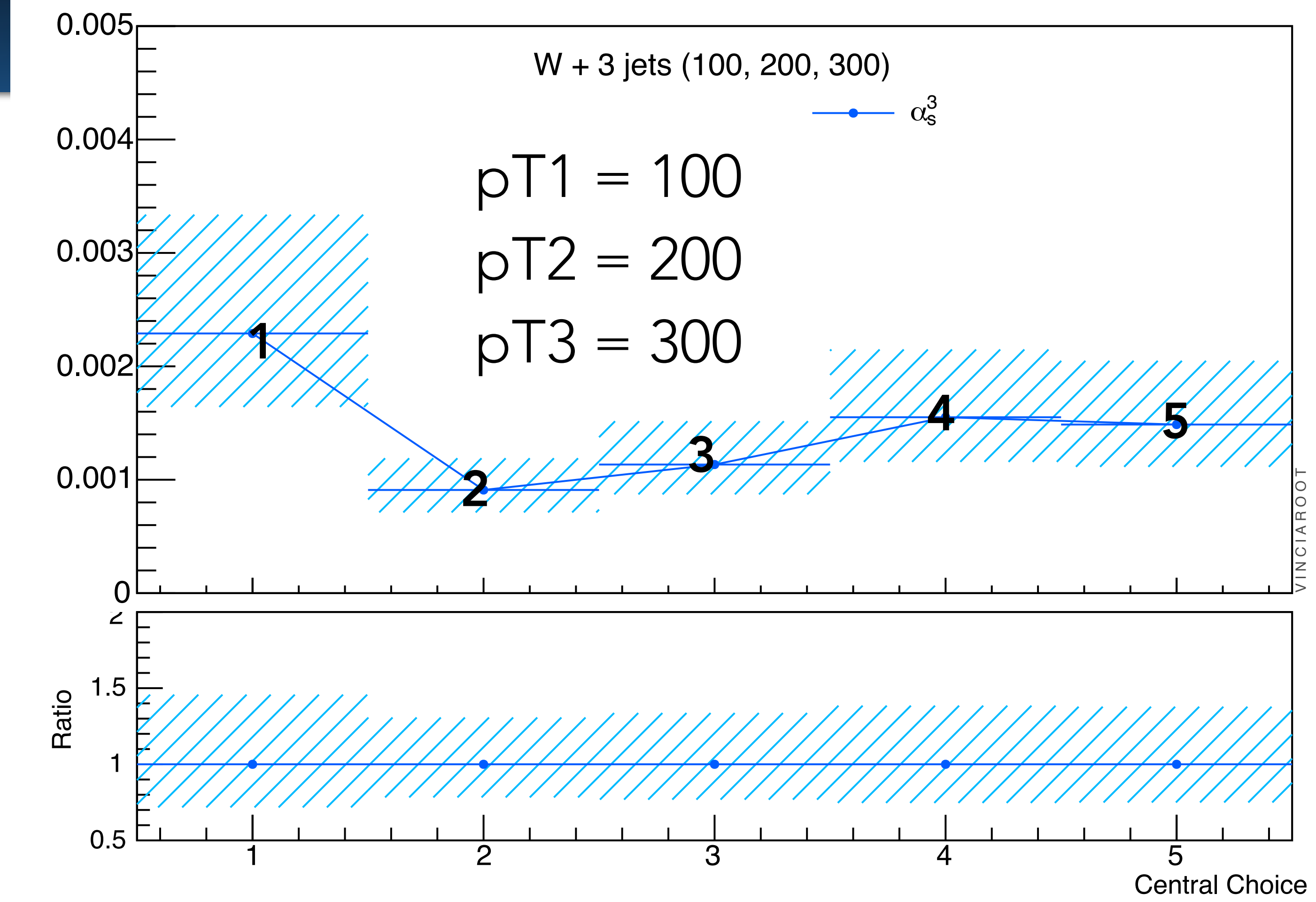
Warning: **Multi-Scale** Problems

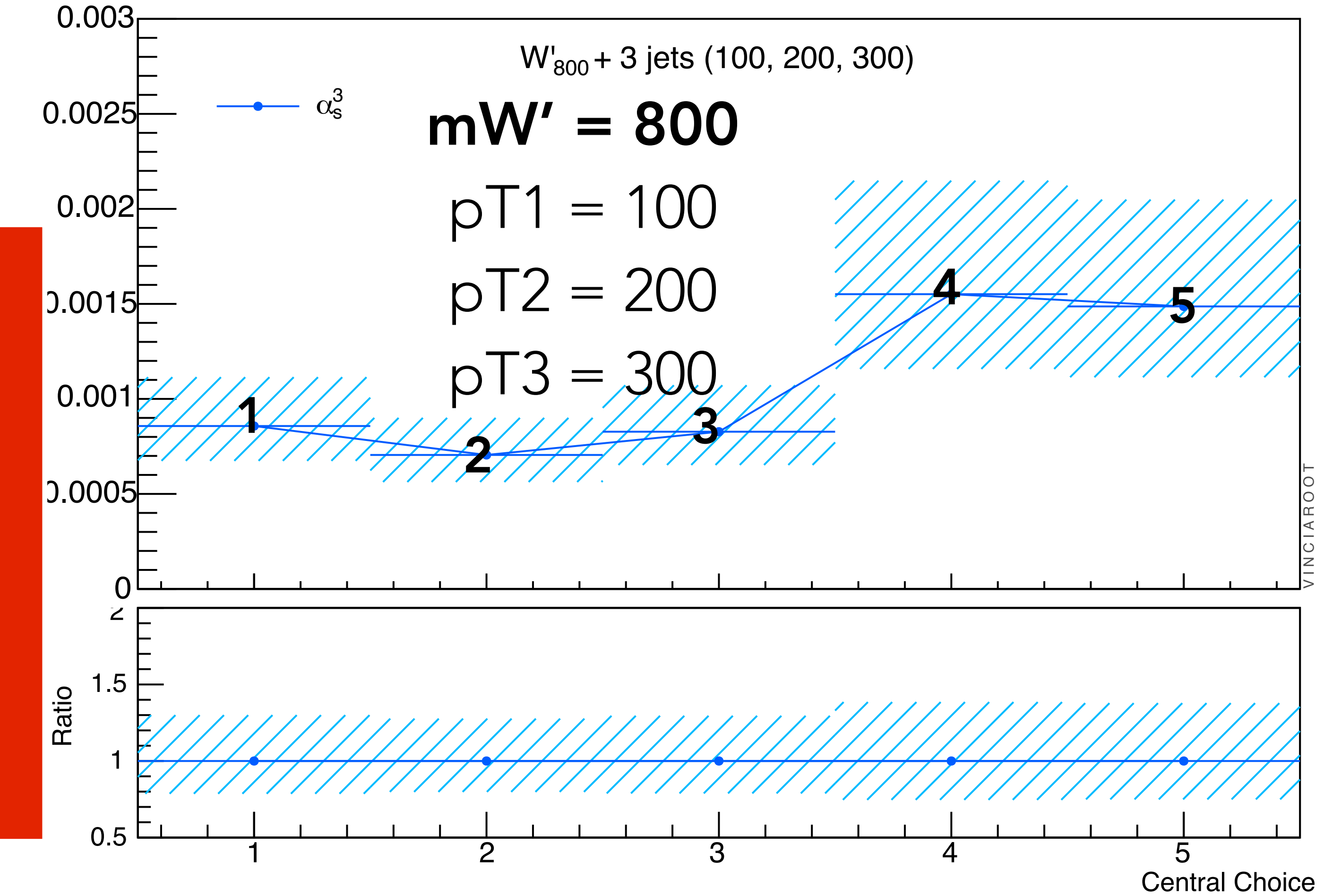
Example: pp → W + 3 jets

Some possible choices for μ_R

1: m_W
2: $m_W + \sum |p_\perp|$
3: as for **2** but summed quadratically
4: Geometric mean $p_\perp$ (~shower)
5: Arithmetic mean $p_\perp$







If you have multiple QCD scales

Variation of single central μ_R by simple factor 2 in each direction **not exhaustive!**

Also consider functional dependence on **each scale in the problem** (+ N$^{(n)}$LO → some compensation)

Cross Sections at Fixed Order in α_s

Now want to compute the distribution of some observable: O

In $\underbrace{\text{“inclusive X production”}}_{\text{X + anything}}$ (suppressing PDF factors)

Sum over identical amplitudes, then square

Momentum configuration

Fixed Order (All Orders)

$$\left.\frac{d\sigma}{d\mathcal{O}}\right|_{\mathrm{ME}} = \sum_{k=0} \int \mathrm{d}\Phi_{X+k} \left| \sum_{\ell=0} M_{X+k}^{(\ell)} \right|^2 \delta\left(\mathcal{O} - \mathcal{O}(\{p\}_{X+k})\right)$$

Phase Space

Cross Section differentially in O

Sum over “anything” ≈ legs

Matrix Elements for X+k at (ℓ) loops

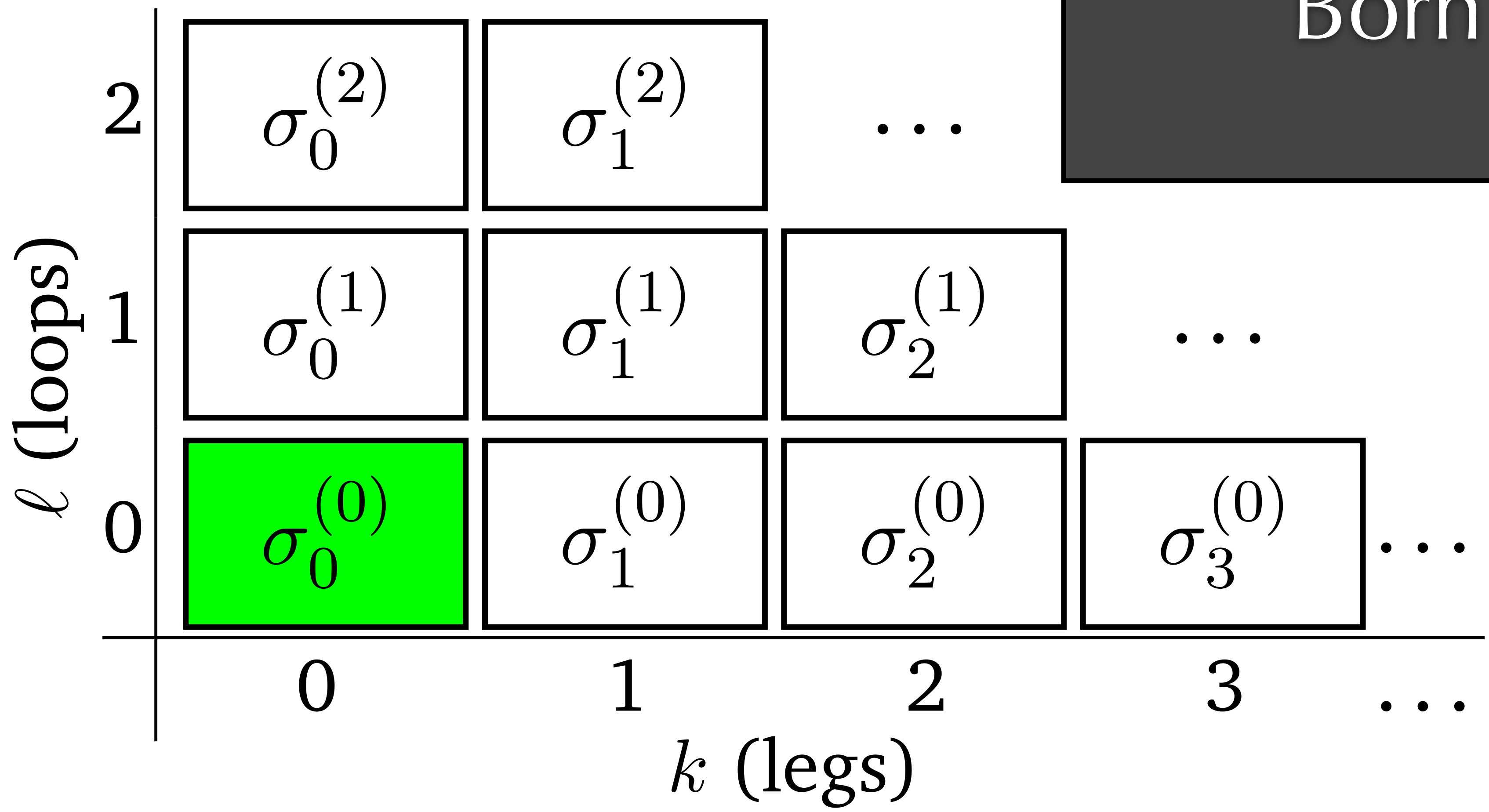
Evaluate observable → differential in O

Truncate at $k = 0, \ell = 0$,

→ **Born Level = First Term**

Lowest order at which X happens

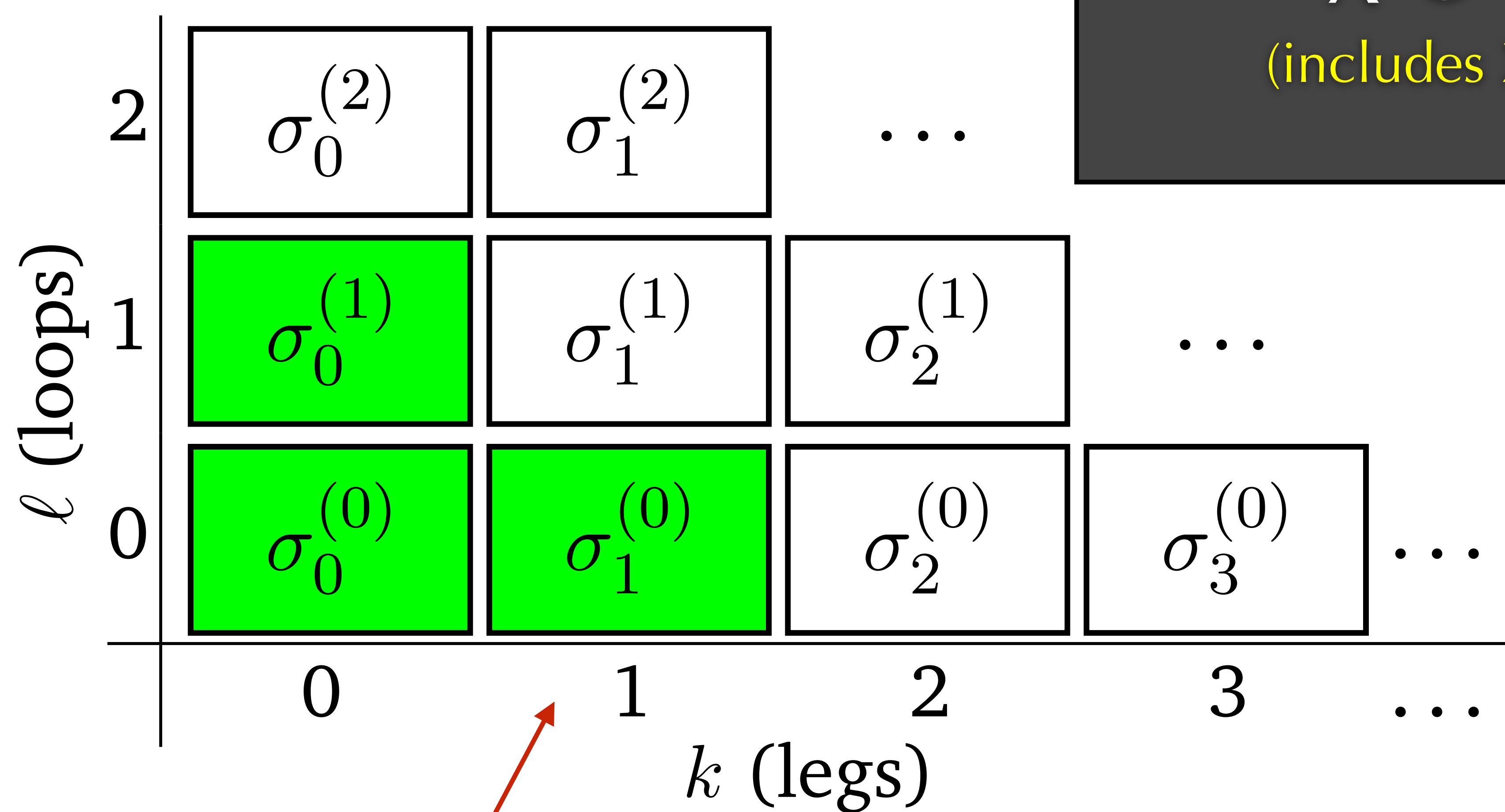
Another representation



Born

Max Born
(1882-1970)
Nobel Prize 1954

Another representation



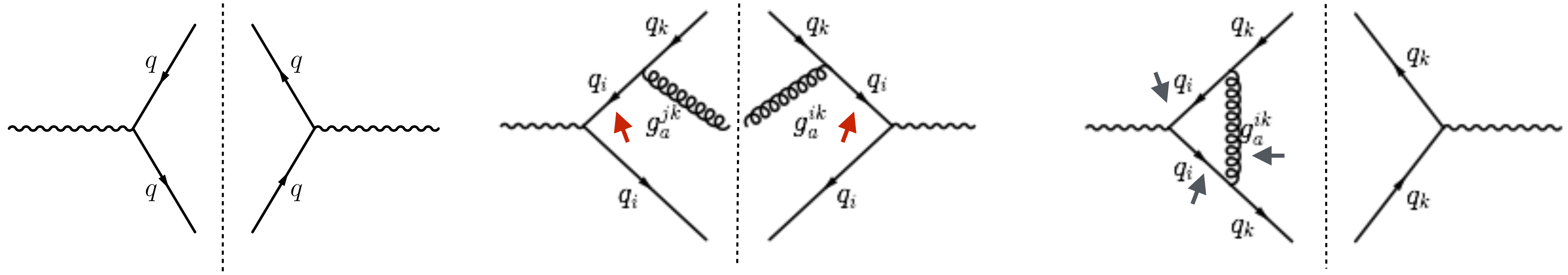
X @ NLO

(includes X+1 @ LO)

Note: (X+1)-jet observables will of course only be correct to LO

NLO:

(note: not the 1-loop diagram squared)



$X^{(2)}$	$X{+}1^{(2)}$	…
$X^{(1)}$	$X{+}1^{(1)}$	…
Born	$X{+}1^{(0)}$	$X{+}2^{(0)}$

$$\sigma_{\mathrm{X}}^{\mathrm{NLO}} = \int |M_X^{(0)}|^2 + \int |M_{X+1}^{(0)}|^2 + \int 2\mathrm{Re}[M_X^{(1)} M_X^{(0)*}]$$

IR singularities

(from poles of propagators going on shell)

In IR limits, the X+1 final state is **indistinguishable** from the X+0 one*

*for so-called IRC safe observables; more later

Sum over 'degenerate quantum states' (KLN Theorem) ➜ **Singularities cancel** when we include both (complete order):

$$= \sigma_{\mathrm{Born}} + \mathrm{Finite}\left\{\int |M_{X+1}^{(0)}|^2\right\} + \mathrm{Finite}\left\{\int 2\mathrm{Re}[M_X^{(1)} M_X^{(0)*}]\right\}$$

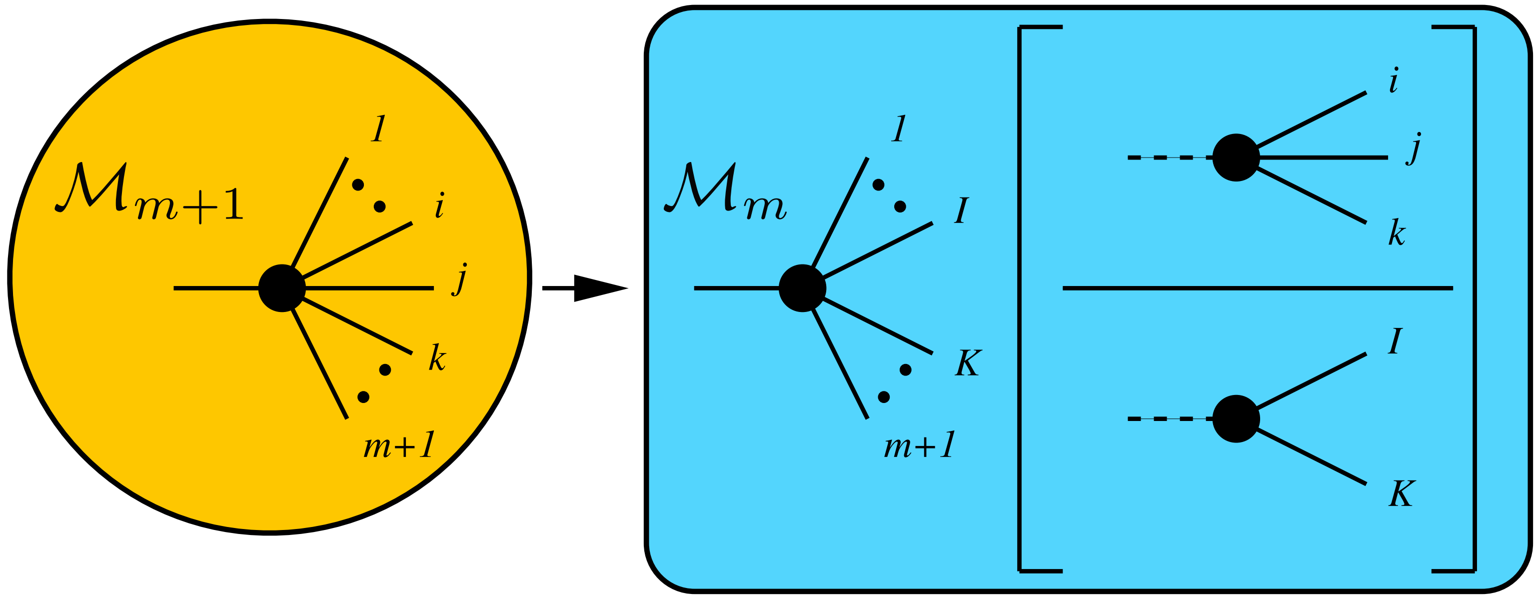
$$\boxed{\sigma_{\mathrm{NLO}}(e^+e^- \to q\bar{q}) = \sigma_{\mathrm{LO}}(e^+e^- \to q\bar{q}) \left(1 + \frac{\alpha_s(E_{\mathrm{CM}})}{\pi} + \mathcal{O}(\alpha_s^2)\right)}$$

How do I get finite{Real} and finite{Virtual} ?

First step: classify IR singularities using universal functions

EXAMPLE: factorization of amplitudes in the **soft** limit

Soft Limit ($E_j \to 0$):



$$|\mathcal{M}_{n+1}(1,\cdots,i,j,k,\cdots,n+1)|^2 \xrightarrow{j_g\to 0} g_s^2 \mathcal{C}_{ijk}\, S_{ijk}\, |\mathcal{M}_n(1,\cdots,i,k,\cdots,n+1)|^2$$

Universal "Soft Eikonal"

$$S_{ijk}(m_I,m_K) = \frac{2s_{ik}}{s_{ij}s_{jk}} - \frac{2m_I^2}{s_{ij}^2} - \frac{2m_K^2}{s_{jk}^2} \qquad s_{ij} \equiv 2p_i \cdot p_j$$

More about this function on next slide & in the next lecture

The Subtraction Idea

Add and subtract IR limits (SOFT and COLLINEAR)

$$\mathrm{d}\sigma_{NLO} = \underbrace{\int_{\mathrm{d}\Phi_{m+1}} \left(\mathrm{d}\sigma^R_{NLO} - \mathrm{d}\sigma^S_{NLO}\right)}_{\text{Finite by Universality}} + \underbrace{\left[\int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\sigma^S_{NLO} + \int_{\mathrm{d}\Phi_m} \mathrm{d}\sigma^V_{NLO}\right]}_{\text{Finite by KLN}}$$

Dipoles (Catani-Seymour)

Global Antennae (Gehrmann, Gehrmann-de Ridder, Glover)

Sector Antennae (Kosower)

…

Choice of subtraction terms:

Singularities mandated by gauge theory

Non-singular terms: up to you (added and subtracted, so vanish)

$$\frac{|\mathcal{M}(Z^0 \to q_i g_j \bar{q}_k)|^2}{|\mathcal{M}(Z^0 \to q_I \bar{q}_K)|^2} = g_s^2\, 2C_F \left[\overbrace{\frac{2s_{ik}}{s_{ij}s_{jk}}}^{\text{SOFT}} + \frac{1}{s_{IK}} \overbrace{\left(\frac{s_{ij}}{s_{jk}} + \frac{s_{jk}}{s_{ij}} \right)}^{\text{COLLINEAR}} \right]$$

$$\frac{|\mathcal{M}(H^0 \to q_i g_j \bar{q}_k)|^2}{|\mathcal{M}(H^0 \to q_I \bar{q}_K)|^2} = g_s^2\, 2C_F \left[\underbrace{\frac{2s_{ik}}{s_{ij}s_{jk}}}_{\text{SOFT}} + \frac{1}{s_{IK}} \left(\underbrace{\frac{s_{ij}}{s_{jk}} + \frac{s_{jk}}{s_{ij}}}_{\text{COLLINEAR}} + \underbrace{2}_{\text{+F}} \right) \right]$$

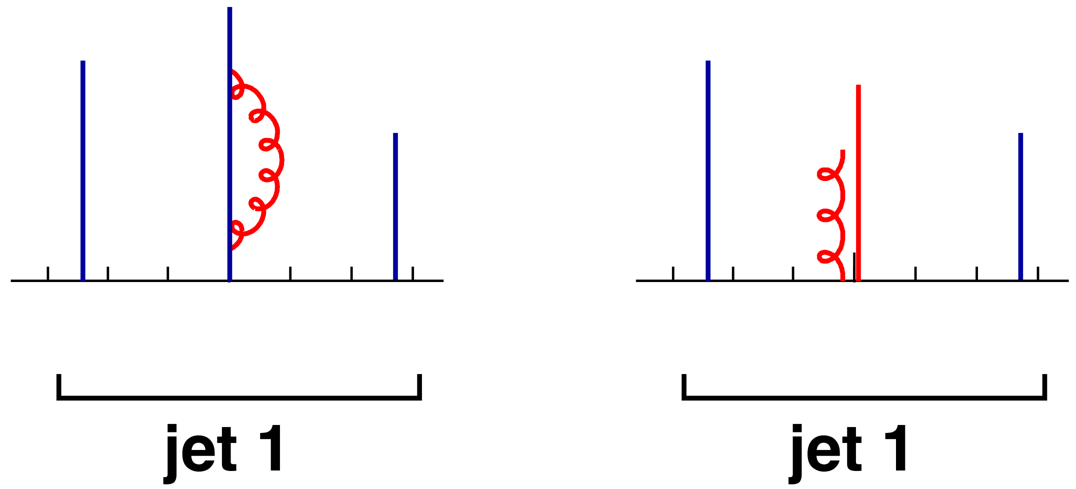
Note on Observables

(example by G. Salam)

Not all observables can be computed perturbatively:

Collinear Safe

Virtual and Real go into **same bins!**



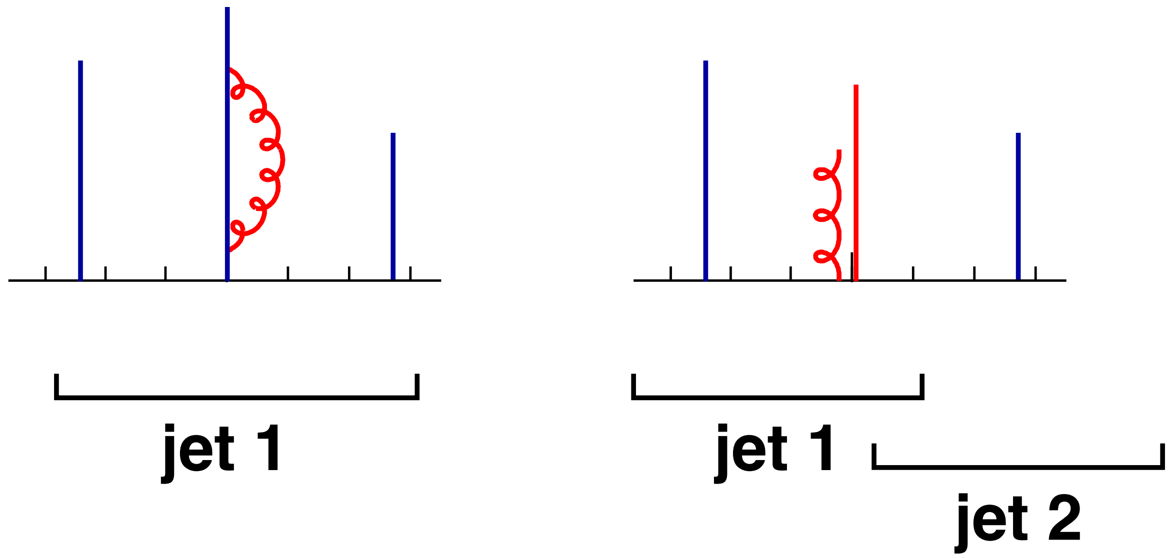
$\alpha_s^n \times (-\infty)$ $\quad$ $\alpha_s^n \times (+\infty)$

Infinities cancel

(KLN: ‘degenerate states’)

Collinear Unsafe

Virtual and Real go into **different bins!**



$\alpha_s^n \times (-\infty)$ $\quad$ $\alpha_s^n \times (+\infty)$

Infinities do not cancel

Invalidates perturbation theory

Perturbatively Calculable $\Longleftrightarrow$ "Infrared and Collinear Safe"

Definition: an observable is infrared and collinear safe if it is *insensitive* to

SOFT radiation:

Adding any number of infinitely *soft* particles (zero-energy) should not change the value of the observable

COLLINEAR radiation:

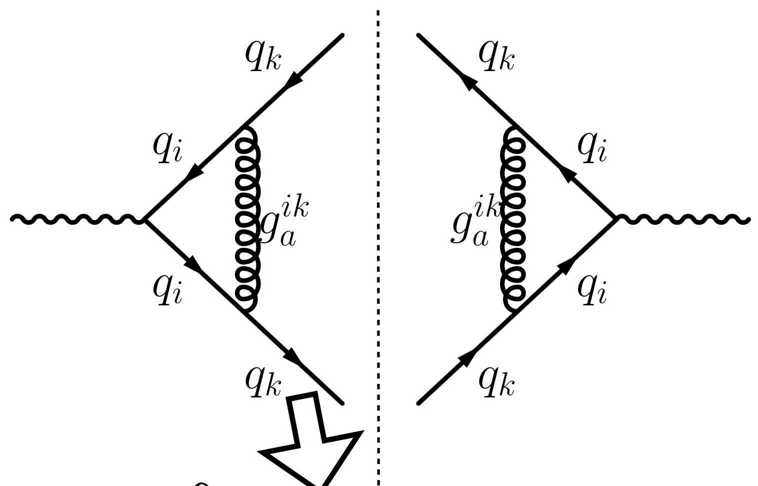
Splitting an existing particle up into two *comoving* ones (conserving the total momentum and energy) should not change the value of the observable

More on this in Lecture 2

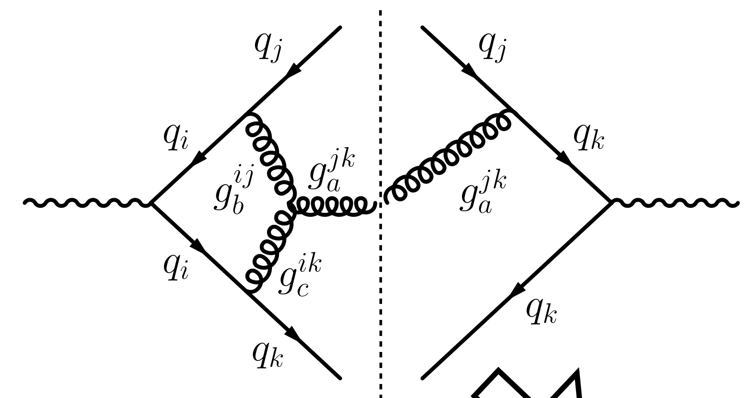
Structure of an NNLO calculation

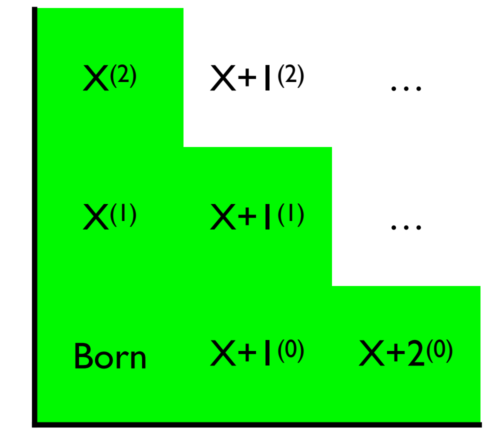
At Next-to-Next-to-Leading Order (NNLO):

1-Loop × 1-Loop



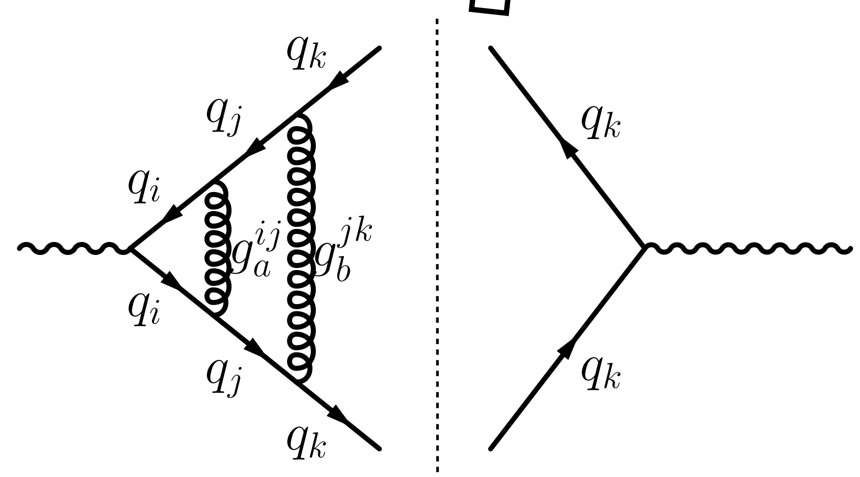
1-Loop × Real for (X+1)



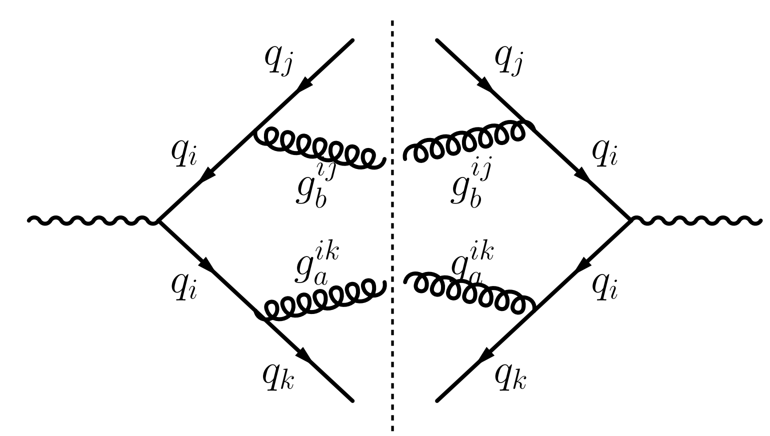


$$\sigma_X^{\mathrm{NNLO}} = \sigma_X^{\mathrm{NLO}} + \int \left(|M_X^{(1)}|^2 + 2\mathrm{Re}[M_X^{(2)} M_X^{(0)*}] \right) + \int 2\mathrm{Re}[M_{X+1}^{(1)} M_{X+1}^{(0)*}] + \int |M_{X+2}^{(0)}|^2$$

Everything we had at NLO



Two-Loop × Born Interference



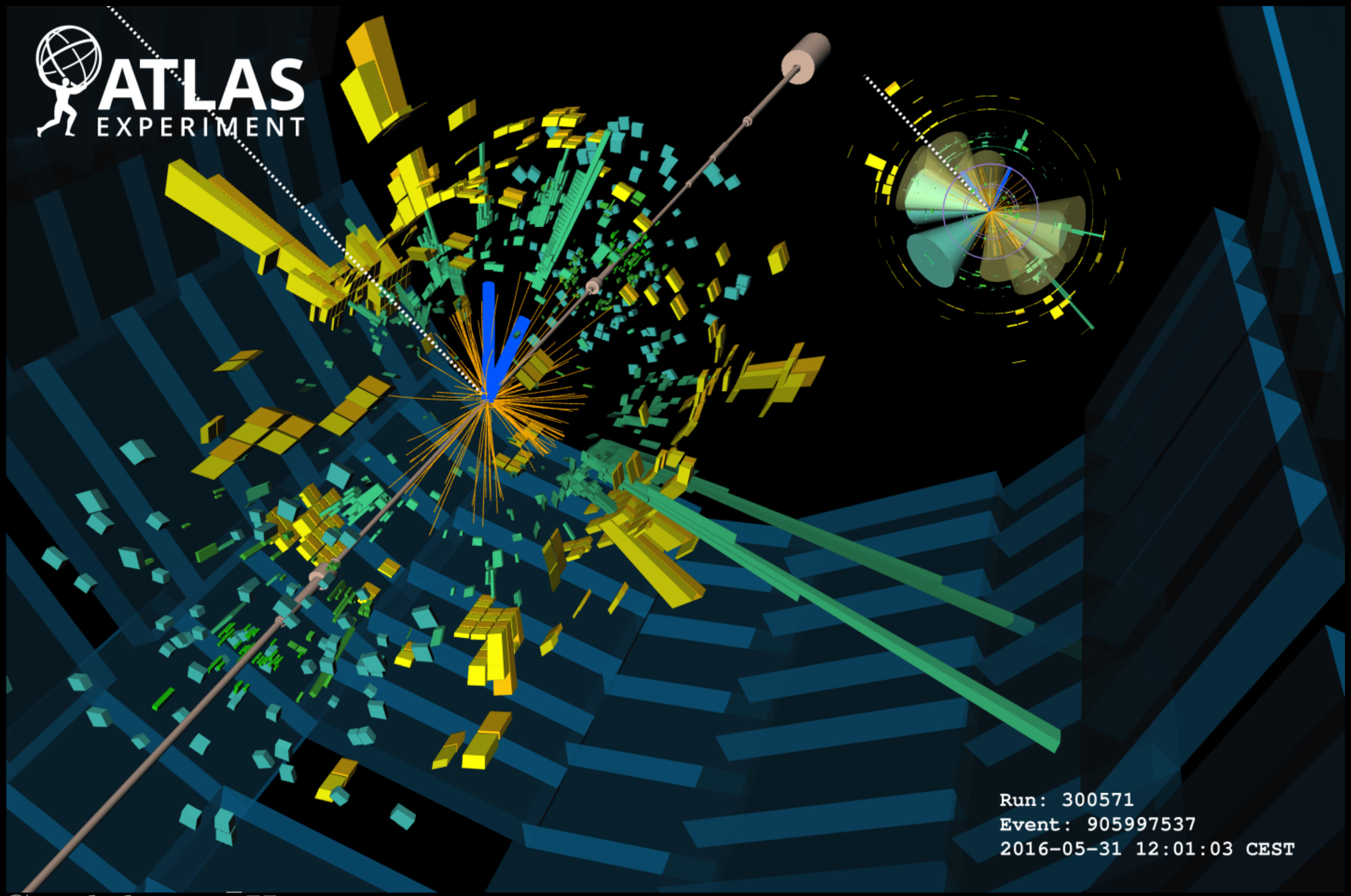
Real × Real for (X+2)

Outlook: dσ/dΩ; how hard can it be?

Approximate all contributing amplitudes for this …

To all orders… then square including interference effects, …

+ non-perturbative effects



Candidate $t\bar{t}H$ event   ATLAS-PHOTO-2016-014-13

… integrate it over a ~300-dimensional phase space

(+ match or exceed statistics of collider that delivers 40 million collisions per second)

Too much for us (today).

Extra Slides

Gell-Mann Matrices

The generators of SU(3) are the ***"Gell-Mann matrices:"***

= the analogs of the SU(2) Pauli matrices

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

(using a pretty "standard" basis choice)

These are (a representation of) the generators of the Non-Abelian group SU(3).
➜ Feynman rules have a Gell-Mann matrix in each quark-gluon vertex. (Normally sum over all.)
There are also ggg and gggg self-interaction vertices. (Absent in QED; no photon self-int.)

Combinations of Colour States

The rules of SU(3) group theory tells us how to combine **colour charges**

Quark + Antiquark :

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$$

Already discussed the octet

The singlet is $\frac{1}{\sqrt{3}} \left| R\bar{R} + G\bar{G} + B\bar{B} \right\rangle$

What does it mean that it is a singlet?

Quark + Quark :

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}$$

The "sextet" includes all the symmetric combinations

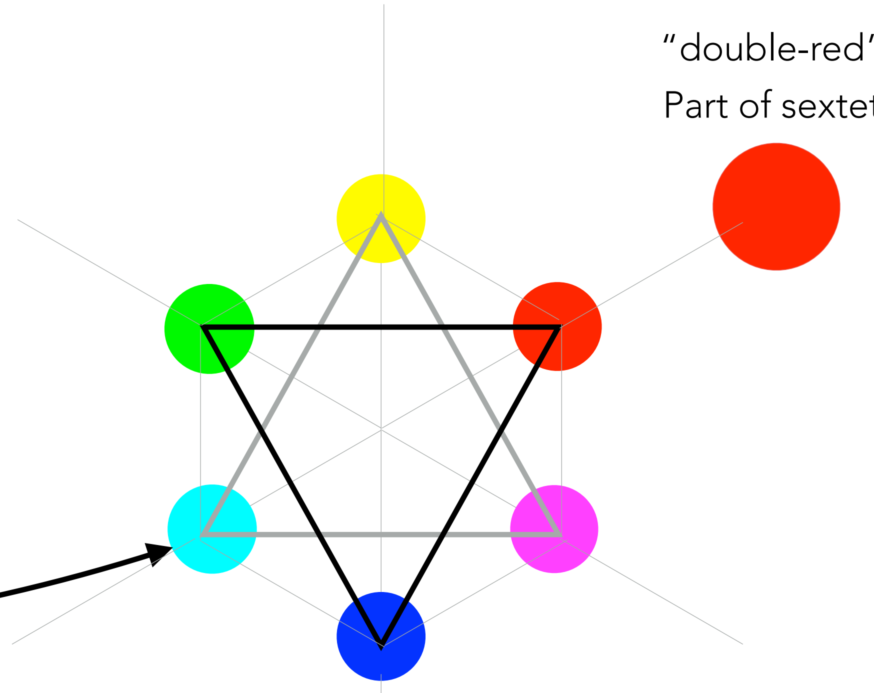
$$|RR\rangle\,, |GG\rangle\,, |BB\rangle\,, |RG + GR\rangle\,, |GB + BG\rangle\,, |BR + RB\rangle$$

The antitriplet includes the antisymmetric combinations

$$|RG - GR\rangle\,, |GB - BG\rangle\,, |BR - RB\rangle$$

Antisymmetrically Combined

E.g., Green+Blue ~ Cyan = antiRed



Interactions in Colour Space

Colour Factors

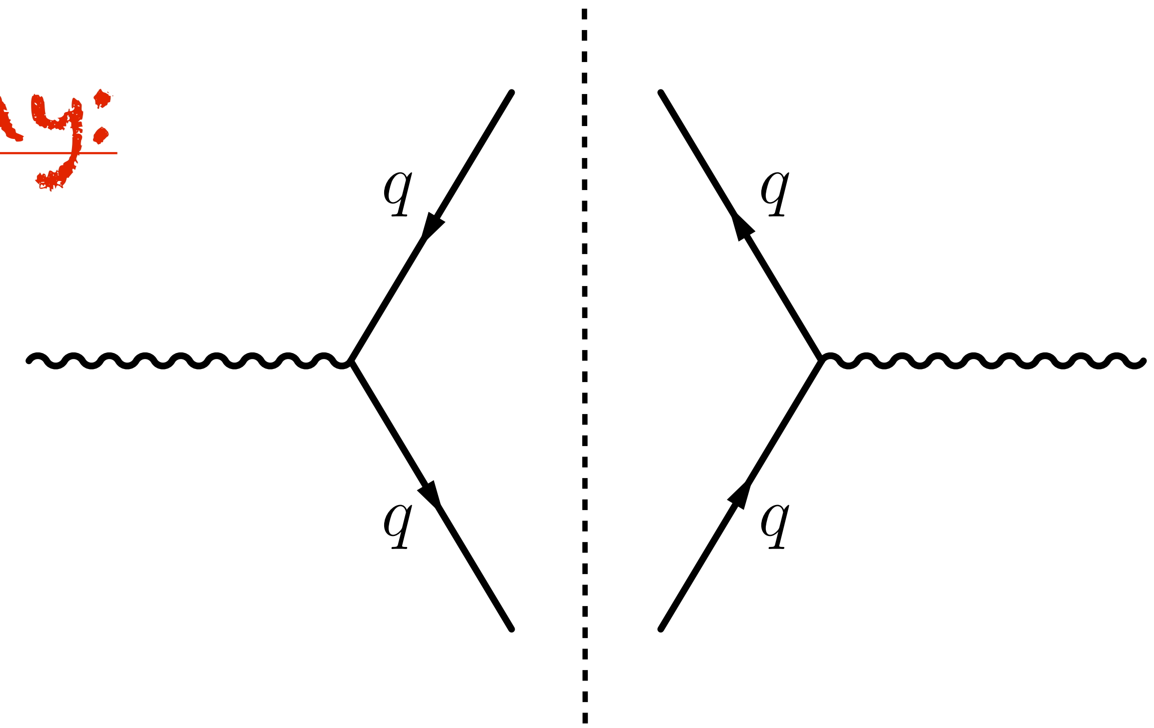
Processes involving coloured particles have a "colour factor".

It counts the enhancement from the sum over colours.

(*average* over incoming colours → can also give suppression)

Z Decay:

$$\sum_{\text{colours}} |M|^2 =$$



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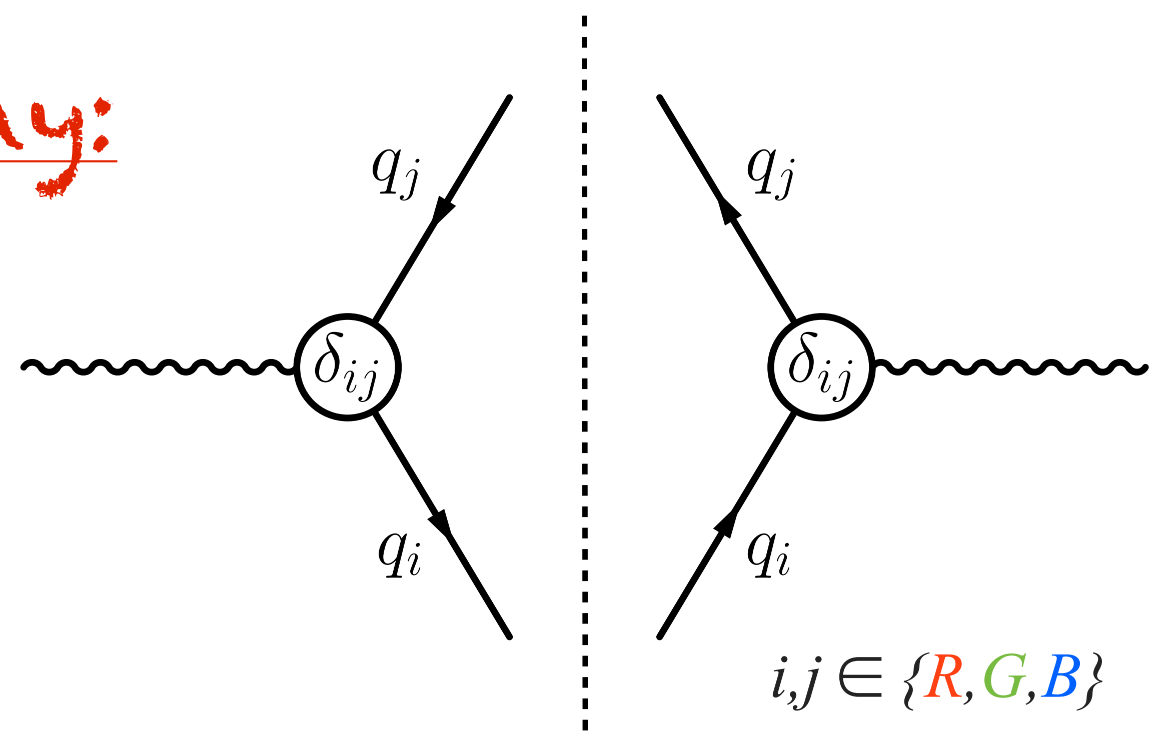


$$\propto \delta_{ij}\delta^*_{ji} = \mathrm{Tr}[\delta_{ij}] = N_C$$

Interactions in Colour Space

Colour Factors

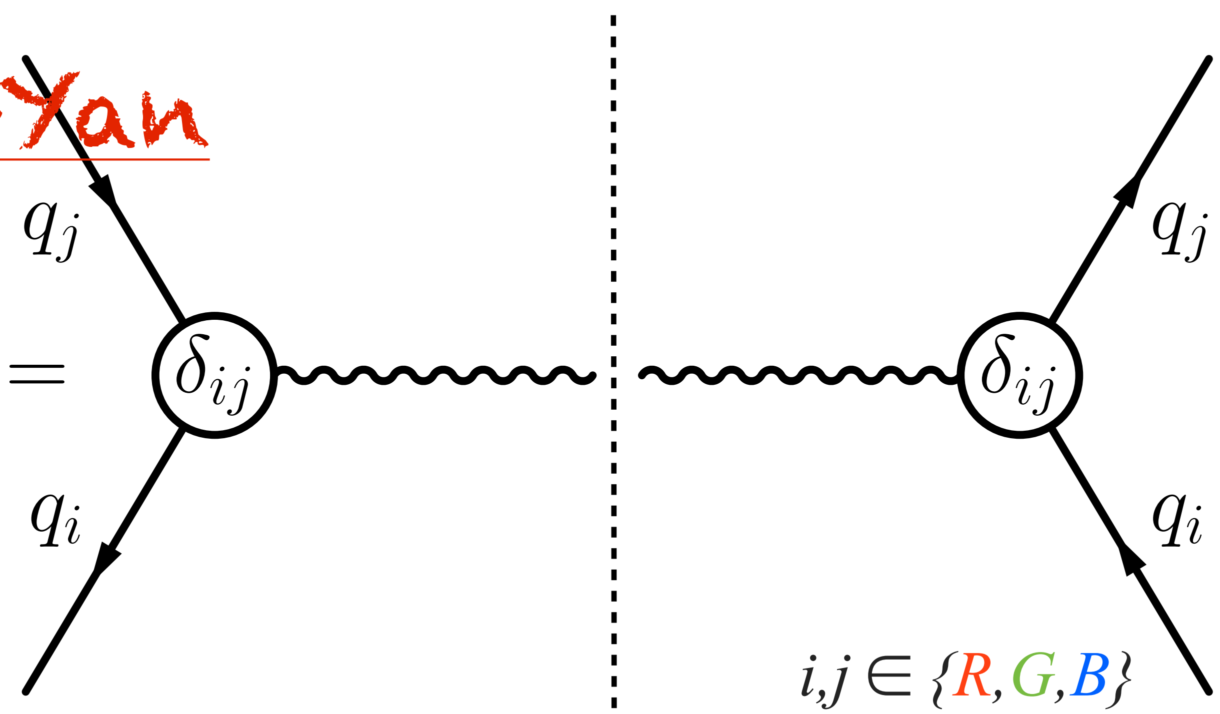
Processes involving coloured particles have a "colour factor".

It counts the enhancement from the sum over colours.

(*average* over incoming colours → can also give suppression)

Drell-Yan

$$\frac{1}{9}\sum_{\text{colours}} |M|^2 =$$



$$\propto \delta_{ij}\delta^*_{ji}\,\frac{1}{N_C^2}$$
$$= \mathrm{Tr}[\delta_{ij}]\,\frac{1}{N_C^2}$$
$$= 1/N_C$$

"Hard" and "Soft"

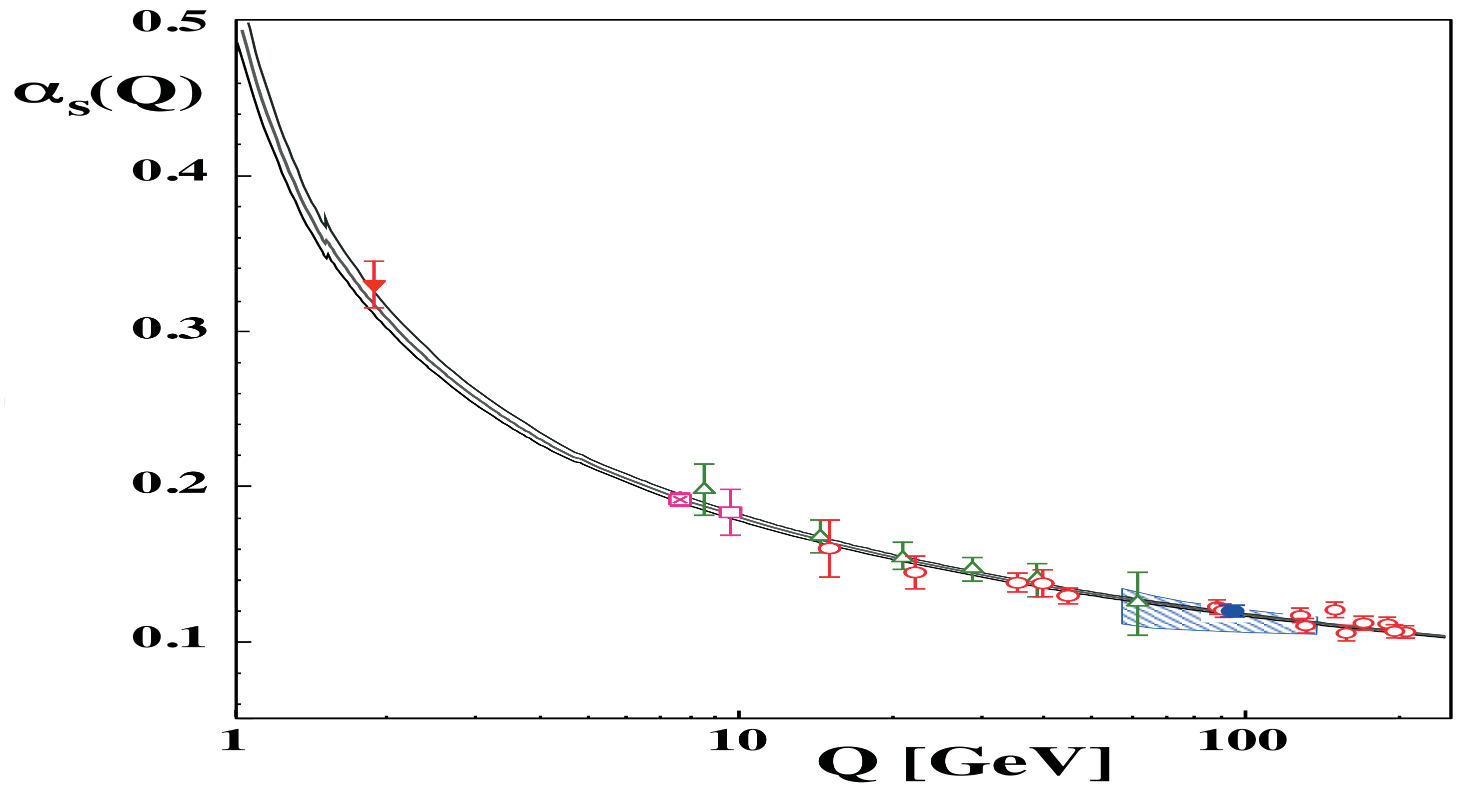
1) In **absolute** terms

"Hard" ~ "perturbative"

Characteristic scale $\gg$ 1 GeV $\implies \alpha_s(Q) \ll 1$

"Soft" ~ "non-perturbative"

Characteristic scale $\lesssim$ 1 GeV



2) In **relative** terms

(more about this tomorrow)

E.g., "the **hard** subprocess" = "the **hardest** subprocess"

A jet with p_T = 30 GeV is **hard** in absolute terms (perturbative) but also **soft relative to** processes at higher scales (say, $t\bar{t}$ production)

Many ways to skin a cat

MCs: get value from: PDG? PDFs? Fits to data (tuning)?

Example (for Final-State Radiation):

SHERPA :

Uses **PDF** or **PDG** value, with **"CMW" translation**

$\alpha_s(m_Z)$ default = **0.118** (pp) or 0.1188 (LEP)

running order: default = **3-loop** (pp) or 2-loop (LEP)

CMW scheme translation: default use ~ $\alpha_s(p_\perp/1.6)$

→ roughly 10% increase in effective value of $\alpha_s(m_Z)$

Will undershoot LEP 3-jet rate by ~ 10% (unless combined with NLO 3-jet ME)

PYTHIA

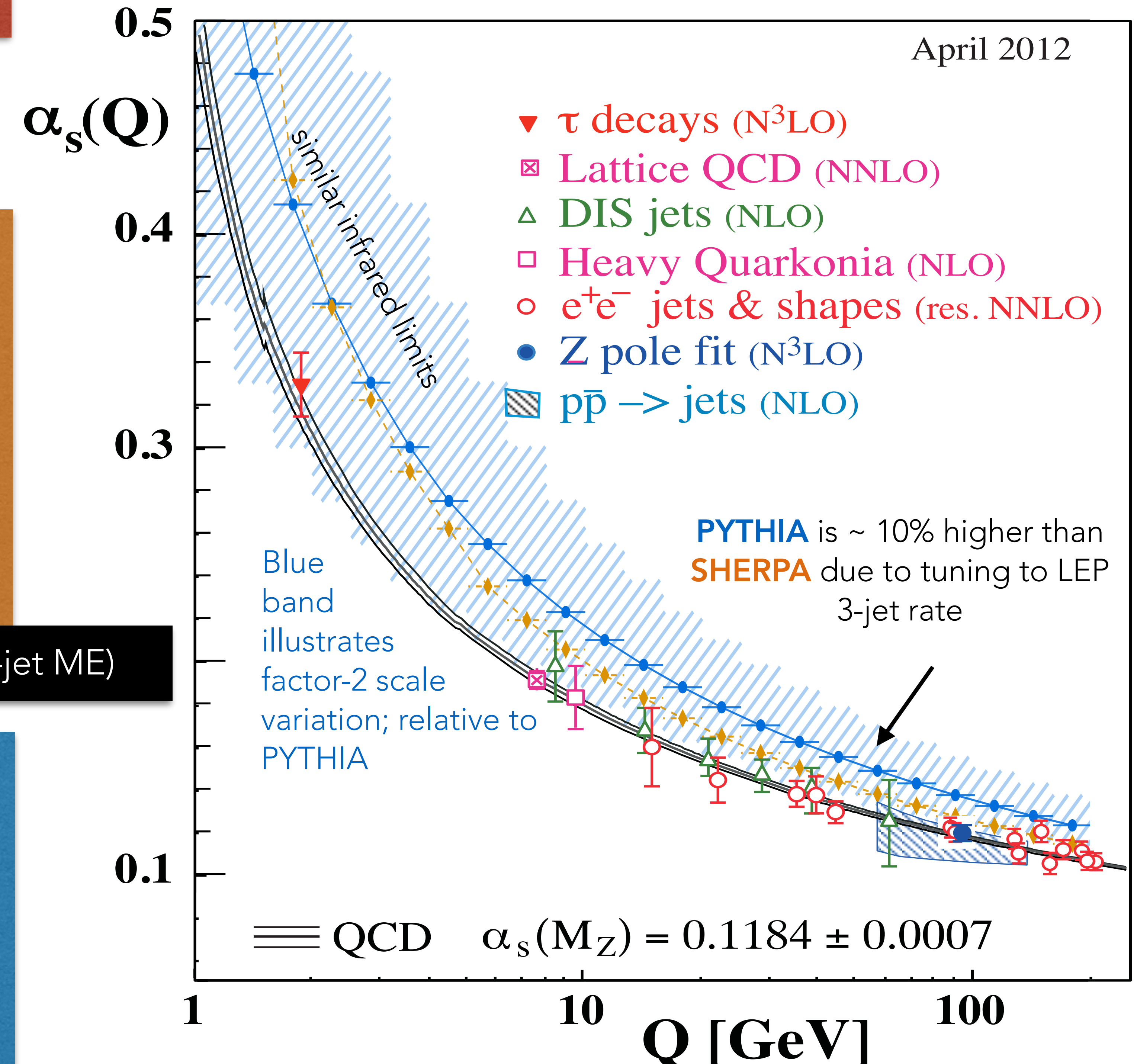
Tuning to **LEP 3-jet rate**; requires ~ 20% increase

TimeShower:alphaSvalue default = **0.1365**

TimeShower:alphaSorder default = **1**

TimeShower:alphaSuseCMW default = **off**

Agrees with LEP 3-jet rate "out of the box"; but no guarantee tuning is universal.

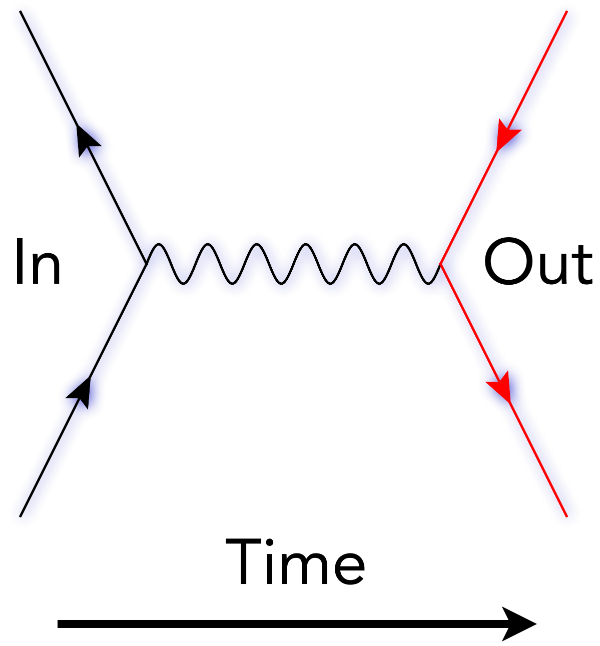


(also note: MC definitions of $Q=p_T$ not identical)

Crossings

$e^+e^- \to \gamma^*/Z \to q\bar{q}$

(Hadronic Z Decay)

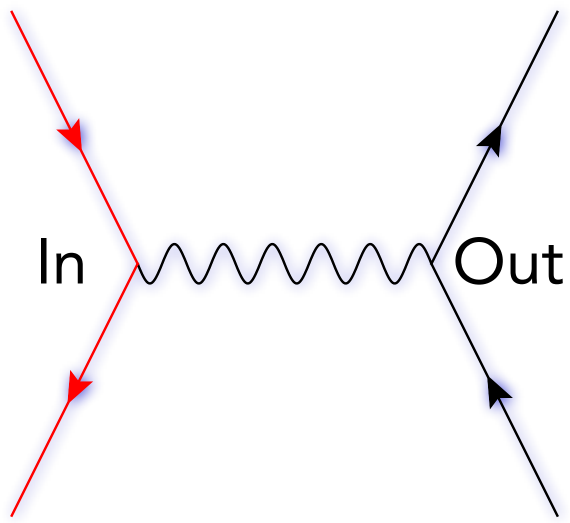


Color Factor:

$$\mathrm{Tr}[\delta_{ij}] = N_C$$

$q\bar{q} \to \gamma^*/Z \to \ell^+\ell^-$

(Drell & Yan, 1970)

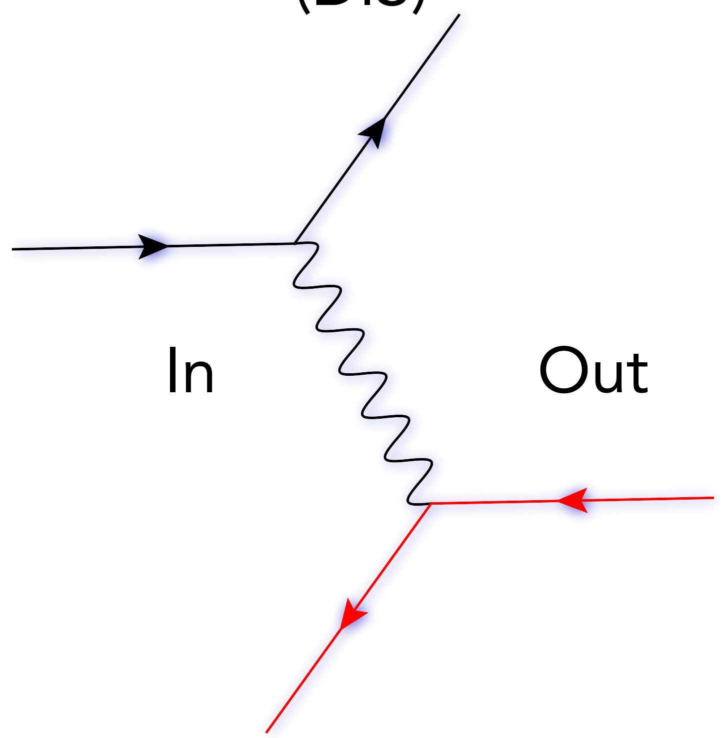


Color Factor:

$$\frac{1}{N_C^2}\mathrm{Tr}[\delta_{ij}] = \frac{1}{N_C}$$

$\ell q \xrightarrow{\gamma^*/Z} \ell q$

(DIS)



Color Factor:

$$\frac{1}{N_C}\mathrm{Tr}[\delta_{ij}] = 1$$

Colour Factors

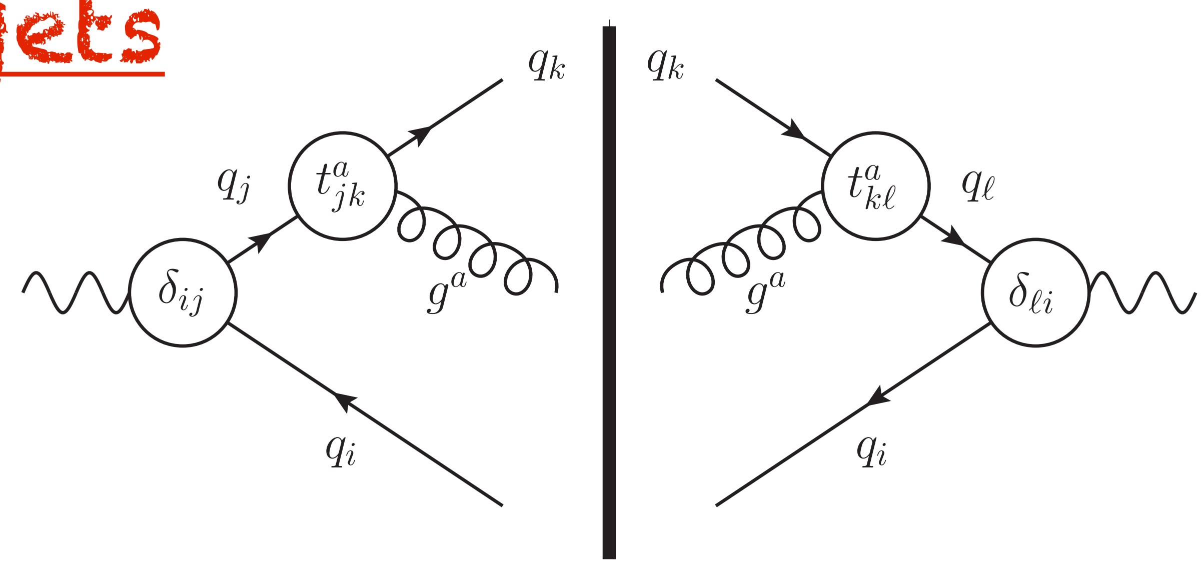
Processes involving coloured particles have a "colour factor".

It counts the enhancement from the sum over colours.

(*average* over incoming colours → can also give suppression)

Z→3 jets

$$\sum_{\text{colours}} |M|^2 =$$



$$\propto \delta_{ij} t^a_{jk} t^a_{k\ell} \delta_{\ell i}$$

$$= \mathrm{Tr}\{t^a t^a\}$$

$$= \frac{1}{2}\mathrm{Tr}\{\delta\} = 4$$

$i,j \in \{R,G,B\}$
$a \in \{1,\ldots,8\}$

Quick Guide to Colour Algebra

Colour factors squared produce traces

Trace Relation

$$\mathrm{Tr}(t^A t^B) = T_R \delta^{AB}\,, \quad T_R = \frac{1}{2}$$

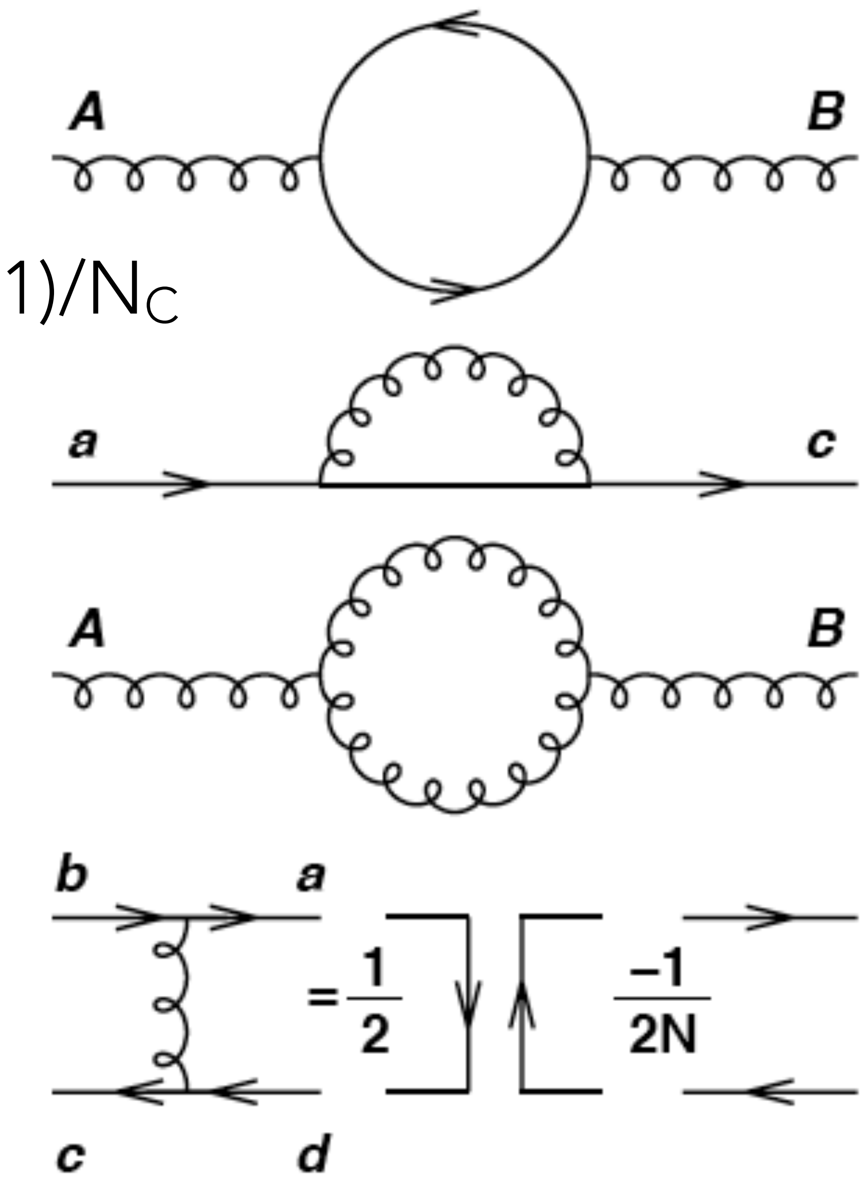
$T_R(N_c^2-1)/N_C$

$$\sum_A t^A_{ab} t^A_{bc} = C_F \delta_{ac}\,, \quad C_F = \frac{N_c^2-1}{2N_c} = \frac{4}{3}$$

$$\sum_{C,D} f^{ACD} f^{BCD} = C_A \delta^{AB}\,, \quad C_A = N_c = 3$$

$$t^A_{ab} t^A_{cd} = \underbrace{\frac{1}{2}}_{T_R}\delta_{bc}\delta_{ad} - \underbrace{\frac{1}{2N_c}}_{T_R/N_C}\delta_{ab}\delta_{cd} \quad \text{(Fierz)}$$

Example Diagram



(from ESHEP lectures by G. Salam)

Real QCD isn't conformal

The coupling runs logarithmically with the energy scale

$$Q^2 \frac{\partial \alpha_s}{\partial Q^2} = \beta(\alpha_s) \qquad \beta(\alpha_s) = -\alpha_s^2 (b_0 + b_1 \alpha_s + b_2 \alpha_s^2 + \ldots) \; ,$$

$$b_0 = \frac{11 C_A - 2 n_f}{12\pi} \qquad b_1 = \frac{17 C_A^2 - 5 C_A n_f - 3 C_F n_f}{24\pi^2} = \frac{153 - 19 n_f}{24\pi^2}$$

1-Loop β function coefficient

2-Loop β function coefficient

$$b_2 = \frac{2857 - 5033 n_f + 325 n_f^2}{128\pi^3}$$

$$b_3 = \text{known}$$

Asymptotic freedom in the ultraviolet

Confinement (IR slavery?) in the infrared

Multi-Scale Exercise

Skands, TASI Lectures, arXiv:1207.2389

If needed, can convert from multi-scale to single-scale

$$\alpha_s(\mu_1)\alpha_s(\mu_2)\cdots\alpha_s(\mu_n) \quad = \quad \prod_{i=1}^{n}\alpha_s(\mu)\left(1 + b_0\,\alpha_s \ln\left(\frac{\mu^2}{\mu_i^2}\right) + \mathcal{O}(\alpha_s^2)\right)$$

$$= \quad \alpha_s^n(\mu)\left(1 + b_0\,\alpha_s \ln\left(\frac{\mu^{2n}}{\mu_1^2\mu_2^2\cdots\mu_n^2}\right) + \mathcal{O}(\alpha_s^2)\right)$$

by taking geometric mean of scales